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Nonlinear surface waves in closed basins

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The Lagrangian and Hamiltonian for nonlinear gravity waves in a cylindrical basin are constructed in terms of the generalized co-ordinates of the free-surface displacement, $\{q_n(t)\} \equiv \mathbf{q}$, thereby reducing the continuum-mechanics problem to one in classical mechanics. This requires a preliminary description, in terms of $\mathbf{q}$, of the fluid motion beneath the free surface, which kinematical boundary-value problem is solved through a variational formulation and the truncation and inversion of an infinite matrix. The results are applied to weakly coupled oscillations, using the time-averaged Lagrangian, and to resonantly coupled oscillations, using Poincaré's action–angle formulation. The general formulation provides for excitation through either horizontal or vertical translation of the basin and for dissipation. Detailed results are given for free and forced oscillations of two, resonantly coupled degrees of freedom.

1. Introduction

The primary end of the following study is the construction and implementation of the Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}})$ for nonlinear gravity waves in a cylindrical basin, where $\mathbf{q} \equiv \{q_n(t)\}$ is a column matrix of the generalized co-ordinates defined by the expansion of the free-surface displacement in a complete set of eigenfunctions (normal modes) that are determined by the linear theory.

The construction of L requires the determination of $\{\phi_n(t)\} \equiv \boldsymbol{\phi}$, the generalized co-ordinates for the fluid motion, in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$. We obtain the solution of this kinematical problem in § 2 in the form $\boldsymbol{\phi}(\mathbf{q}, \dot{\mathbf{q}}) = \boldsymbol{l}(\mathbf{q})\,\dot{\mathbf{q}}$, where $\boldsymbol{l}(\mathbf{q})$ is a square matrix that may be expanded in powers of $\mathbf{q}$. This result, which reflects the linearity of the *kinematical* boundary-value problem for prescribed $\mathbf{q}$ (nonlinearity in the velocity enters only through the *dynamical* condition of uniform pressure at the free surface), is formally exact; however, the explicit determination of $\boldsymbol{l}$ requires the inversion of an antecedent matrix equation, which in turn requires truncation at some finite power of $\mathbf{q}$.

We construct $L(\mathbf{q}, \dot{\mathbf{q}})$ in § 3 and invoke Hamilton's principle to obtain the second-order differential equations for $\{q_n(t)\}$. We then go on, in § 4, to obtain the momentum matrix $\mathbf{p} = \partial L/\partial \dot{\mathbf{q}}$ and the Hamiltonian $H(\mathbf{q}, \mathbf{p})$. Capillary waves could be accommodated by incorporating the free-surface energy in both H and L but would require a description of the variation of the surface tension for waves of finite amplitude.

The perturbation pressure, which does not enter the formulations of §§ 3 and 4 (the explicit invocation of the dynamical free-surface condition is replaced by Hamilton's principle), is calculated in § 5 and used to construct the Lagrangian $L_*(\mathbf{q}, \boldsymbol{\phi})$, following Luke (1967) and Whitham (1974, § 13.2). We find that the first-order differential equations for $\mathbf{q}$ and $\boldsymbol{\phi}$ implied by L_* exhibit a strong structural similarity to those for $\mathbf{q}$ and $\mathbf{p}$ implied by H. The two formulations are related by the transformation $\mathbf{p} = \boldsymbol{d}(\mathbf{q})\,\boldsymbol{\phi}$, where the square matrix $\boldsymbol{d}$ is determined by the kinematical boundary-value problem of § 2.

The availability of either L or H reduces the water-wave problem to an equivalent problem in classical mechanics, which then may be attacked by the elegant methods developed by Hamilton and his successors – notably Poincaré (1892–99) – and, in the present century, by the Russian school of nonlinear mechanics (see Minorsky 1947). The resulting formulations are relatively terse and are naturally expressed in terms of universal parameters that are superficially independent of the particular basin cross-section and eigenfunctions.

As a first example, we consider (in § 6) weak coupling of the normal modes and use the average Lagrangian to determine all amplitudes to second order (in the amplitude of the perturbed mode) and the corresponding corrections to the natural frequencies. This problem goes back to Rayleigh (1915), who obtained results for two-dimensional, deep-water waves, and has been solved by Tadjbakhsh & Keller (1960) for two-dimensional, by Verma & Keller (1962) for rectangular, and by Mack (1962) for axisymmetric basins of finite depth. The present results are applicable to basins of arbitrary cross-section and finite depth and point to certain errors and oversights in some of the earlier work.

The power and simplicity of procedures based on the average Lagrangian are perhaps most apparent (in the present context) in the treatment of resonant interactions (cf. Whitham 1967; Simmons 1969); in particular, it leads naturally to the determination of integral (or *adiabatic*) invariants. The efficient construction of these invariants for free oscillations is expedited by starting from the Hamiltonian, rather than the Lagrangian, and invoking Poincaré's (1892, § 6) transformation to *action* and *angle* variables. If only two modes are resonantly coupled, this procedure leads directly to two invariants (one of which is simply H by virtue of conservation of energy) and permits the formal reduction of the problem to quadrature (cf. Whittaker 1944, §§ 193ff.). We carry this procedure through in some detail (in § 7) for the simplest case, in which: the natural frequencies of the two modes stand approximately in the ratio 2:1, the coupling is quadratic, and the solution can be expressed in terms of elliptic functions. This problem has been widely studied in various physical contexts [see Rott (1970) for a delightful mechanical example and references to other examples], but the present discussion appears to contain some novel elements, at least in the context of water waves.†

We remark that the coupling is cubic, and the analysis is correspondingly more

† The general problem of resonant interactions goes back at least to the nineteenth century, when it was studied by Korteweg (1897) in a paper that I have been unable to obtain but which is partially summarized by his student Beth (1913). Its profound implications for mechanics appear to have been recognized originally by Poincaré (1892); see Brillouin (1960).

complicated, if the natural frequencies are approximately equal. The classical prototype is the spherical pendulum, and forced oscillations of the cross-polarized (but otherwise identical) modes in a circular tank have been treated by Hutton (1963) by reduction of the modal equations to those for an equivalent pendulum (Miles 1962).

The practical treatment of resonant interactions in closed basins demands consideration of the forcing mechanism and of dissipation. Perhaps the simplest method of excitation in the laboratory is either horizontal or vertical translation of the basin (cf. Benjamin & Ursell 1954; Chester 1968; Chester & Bones 1968); in any event, it is the simplest for theoretical treatment, and we incorporate it in the formulations of §§3–5. Horizontal translation enters the problem through an appropriate generalized force, whereas vertical translation (or, more precisely, acceleration) enters as a component of the apparent gravitational field. We consider resonant forcing of a pair of resonantly interacting modes (the configuration of §7), with the forcing frequency approximately equal to that of the dominant mode, in §8.

Dissipation is small for those configurations that permit ready observation of nonlinear wave motion and is typically important for any given mode only in the neighbourhood of resonance. The corresponding term in the equation of motion for the nth mode then is proportional to $\mathscr{L}_n \dot{q}_n$ (see §3), where $\mathscr{L}_n$ is the logarithmic decrement of the normal mode and may comprise the effects of boundary-layer friction, surface contamination and capillary hysteresis (Miles 1967).

It seems appropriate to remark that, although the results presented here stem ultimately from classical mechanics, they were stimulated by Whitham's (1967, 1974) work on nonlinear dispersive waves on an unbounded surface. The essential distinction between these waves and those considered here is, of course, that between continuous and discrete spectra.

2. Kinematical problem

We consider irrotational gravity waves in an inviscid liquid of density ρ that fills a rigid cylindrical basin B of cross-section S to a quiescent depth d.† Let $\mathbf{x}$ and y be horizontal and vertical co-ordinates in a reference frame fixed in B, with $y = \eta(\mathbf{x}, t)$ at the free surface and $y = -d$ at the bottom, $\mathbf{n}$ the outwardly directed normal to the fluid boundary, and $\phi(\mathbf{x}, y, t)$ the relative velocity potential ($\nabla\phi =$ fluid velocity relative to B). The kinematical boundary-value problem then is described by

$$\nabla^2\phi = 0 \quad (\mathbf{x} \quad \text{in} \quad S, \quad -d < y < \eta), \tag{2.1}$$

$$\mathbf{n}\,.\,\nabla\phi = 0 \quad \text{on} \quad B, \quad \phi_y - \nabla\eta\,.\,\nabla\phi = \eta_t \quad \text{on} \quad y = \eta, \tag{2.2 a, b}$$

which may be derived by requiring the integral

$$SI = \frac{1}{2}\iiint (\nabla\phi)^2\, dS\, dy - \iint (\phi)_{y=\eta}\, \eta_t\, dS \tag{2.3}$$

† The formal development that follows, in particular, (2.8)–(2.11), remains valid for variable depth, but the explicit results (2.12)–(2.19), (3.3), (4.4) and (5.4) depend on the separation of variables that is implicit in (2.5)–(2.7).

to be stationary with respect to the variation $\delta\phi$ for prescribed η (Dirichlet's principle; cf. Serrin 1959).

Now suppose that

$$\eta(\mathbf{x},t) = q_n(t)\,\psi_n(\mathbf{x}), \quad \phi(\mathbf{x},y,t) = \phi_n(t)\,\chi_n(\mathbf{x},y), \tag{2.4a, b}$$

where q_n and ϕ_n are generalized co-ordinates, $\{\psi_n\}$ and $\{\chi_n\}$ are the eigenfunctions (normal modes in the linear approximation) determined from (2.1) and (2.2a) according to

$$(\nabla^2+k_n^2)\,\psi_n = 0, \quad \mathbf{n}.\nabla\psi_n = 0 \quad \text{on} \quad \partial S, \quad k_n = |\mathbf{k}_n|, \tag{2.5a, b, c}$$

$$\iint \psi_m\psi_n\,dS = \delta_{mn}S \quad \left(\delta_{mn} = \frac{1}{0}, \quad m \mathbin{\substack{=\\ \neq}} n\right) \tag{2.6}$$

and
$$\chi_n(\mathbf{x},y) = \psi_n(\mathbf{x})\operatorname{sech}k_n d\cosh k_n(y+d) \quad (n \text{ not summed}), \tag{2.7}$$

and repeated dummy indices are summed over the complete set of eigenfunctions except as noted. We remark that $\{k_0, \psi_0, \chi_0\} \equiv \{0, 1, 1\}$ is a non-trivial member of the complete set for the expansion of ϕ; however, it does not enter the kinematical problem, for which $q_0 = 0$ may be inferred directly from the constraint of constant volume.

Substituting (2.4) into (2.3) yields

$$I = \tfrac{1}{2}\mathscr{k}_{mn}\phi_m\phi_n - \mathscr{d}_{mn}\dot{q}_m\phi_n \tag{2.8a}$$

$$\equiv \tfrac{1}{2}\boldsymbol{\phi}'\boldsymbol{k}\boldsymbol{\phi} - \dot{\mathbf{q}}'\boldsymbol{d}\boldsymbol{\phi}, \tag{2.8b}$$

where

$$\mathscr{d}_{mn} = S^{-1}\iint (\chi_n)_{y=\eta}\,\psi_m\,dS, \quad \mathscr{k}_{mn} = S^{-1}\iint dS\int_{-d}^{\eta}\nabla\chi_m.\nabla\chi_n\,dy, \tag{2.9a, b}$$

$\boldsymbol{\phi} \equiv \{\phi_n\}$ and $\dot{\mathbf{q}} \equiv \{\dot{q}_n\}$ are column matrices, $\boldsymbol{\phi}'$ and $\dot{\mathbf{q}}'$ are their transposes (row matrices), $\boldsymbol{k} \equiv [\mathscr{k}_{mn}]$ is a symmetric square matrix with the dimensions of wave-number (inverse length), and $\boldsymbol{d} \equiv [\mathscr{d}_{mn}]$ is an asymmetric, dimensionless, square matrix. Invoking the aforementioned variational principle yields

$$\partial I/\partial\boldsymbol{\phi} = \boldsymbol{k}\boldsymbol{\phi} - \dot{\mathbf{q}}'\boldsymbol{d} = 0, \tag{2.10}$$

from which it follows that

$$\boldsymbol{\phi} = \boldsymbol{k}^{-1}\boldsymbol{d}'\dot{\mathbf{q}} \equiv \boldsymbol{l}\dot{\mathbf{q}}, \tag{2.11}$$

where $\boldsymbol{k}^{-1}$ is the inverse of $\boldsymbol{k}$, and $\boldsymbol{l}$ is an asymmetric square matrix with the dimensions of length.

The Taylor series obtained by substituting (2.4a) and (2.7) into (2.9a, b) and expanding the integrands in powers of η are derived in the appendix. The end results, together with the corresponding series obtained from (2.11), are given by

$$\mathscr{d}_{mn} = \delta_{mn} + C_{lmn}\mathscr{k}_n q_l + \tfrac{1}{2}C_{jlmn}k_n^2 q_j q_l + \ldots, \tag{2.12}$$

$$\mathscr{k}_{mn} = \delta_{mn}\mathscr{k}_m + (C_{lmn}\mathscr{k}_m\mathscr{k}_n + D_{lmn})\,q_l + \tfrac{1}{2}[C_{jlmn}(\mathscr{k}_m k_n^2 + \mathscr{k}_n k_m^2) + D_{jlmn}(\mathscr{k}_m + \mathscr{k}_n))]\,q_j q_l + \ldots, \tag{2.13}$$

and

$$\ell_{mn} = \delta_{mn}a_m - D_{lmn}a_m a_n q_l + \tfrac{1}{2}[-C_{jlmn}a_n k_n^2 - D_{jlmn}(a_m + a_n) + 2D_{lni}(C_{ijm} + D_{jmi}a_i a_m)\,a_n]\,q_j q_l + \ldots, \tag{2.14}$$

where
$$\mathscr{k}_n = a_n^{-1} = k_n\tanh k_n d \equiv \omega_n^2/g, \tag{2.15}$$

ω_n is the natural (radian) frequency of the nth normal mode, the indices m and n are not summed in any of (2.12)–(2.15), and

$$C_{lmn} = S^{-1} \iint \psi_l \psi_m \psi_n \, dS, \quad C_{jlmn} = S^{-1} \iint \psi_j \psi_l \psi_m \psi_n \, dS, \ldots \tag{2.16a, b}$$

and

$$D_{lmn} = S^{-1} \iint \psi_l \nabla\psi_m . \nabla\psi_n \, dS, \quad D_{jlmn} = S^{-1} \iint \psi_j \psi_l \nabla\psi_m . \nabla\psi_n \, dS, \ldots \tag{2.17a, b}$$

are correlation integrals. Integrating (2.17a) by parts [see (A 7) and (A 8)] yields

$$D_{lmn} = \tfrac{1}{2} C_{lmn} (k_m^2 + k_n^2 - k_l^2). \tag{2.18}$$

A complete reduction of D_{jlmn} to a form like (2.18) is not possible, but a useful recursion formula is given by (A 9).

The eigenfunctions for two-dimensional waves in the tank $0 < x < \pi/k$ are given by

$$k_n = nk, \quad \psi_n = 2^{\frac{1}{2}} \cos nkx \quad (n = 1, 2, \ldots). \tag{2.19a, b}$$

The corresponding correlation integrals are elementary.

3. Lagrangian formulation

The kinetic energy of the fluid motion described by (2.4) and (2.11) is

$$T = \tfrac{1}{2}\rho \iiint (\nabla\phi)^2 \, dS \, dy = \tfrac{1}{2}\rho S \mathscr{k}_{mn} \phi_m \phi_n = \tfrac{1}{2}\rho S a_{mn} \dot{q}_m \dot{q}_n, \tag{3.1}$$

where

$$\boldsymbol{a} \equiv [a_{mn}] = \boldsymbol{d}\boldsymbol{k}^{-1}\boldsymbol{d}' = \boldsymbol{d}\boldsymbol{l} \tag{3.2}$$

is a symmetric matrix with the dimensions of length. Substituting (2.12) and (2.14) into (3.2) yields

$$a_{mn} = \delta_{mn} a_m + a_{lmn} q_l + \tfrac{1}{2} a_{jlmn} q_j q_l + \ldots, \tag{3.3a}$$

$$a_{lmn} = C_{lmn} - D_{lmn} a_m a_n, \quad a_{jlmn} = -D_{jlmn}(a_m + a_n) + 2D_{jmi} D_{lni} a_i a_m a_n. \tag{3.3b, c}$$

The potential energy of the free-surface displacement is

$$V = \rho \iint dS \int_0^{\eta} [\dot{\mathbf{u}} . \mathbf{x} + (g + \dot{v}) y] \, dy = \rho S(-Q_n q_n + \tfrac{1}{2} \mathscr{g} q_n q_n), \tag{3.4}$$

where $\dot{\mathbf{u}}$ and $\dot{v}$ are the horizontal and vertical accelerations of the basin, g is the gravitational acceleration, the quantity in square brackets is the specific work done against the d'Alembert force (which is conservative in the sense that V is independent of the history of the displacement from $y = 0$ to $y = \eta$), and

$$Q_n = -\dot{\mathbf{u}} . \mathbf{x}_n, \quad \mathbf{x}_n = S^{-1} \iint \mathbf{x} \psi_n \, dS, \quad \mathscr{g} = g + \dot{v}. \tag{3.5a, b, c}$$

The Lagrangian implied by (3.1)–(3.4) after factoring out ρS is

$$L \equiv (\rho S)^{-1} (T - V) = \tfrac{1}{2} a_{mn} \dot{q}_m \dot{q}_n - \tfrac{1}{2} \mathscr{g} q_n q_n + Q_n q_n. \tag{3.6}$$

Invoking Lagrange's equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_n}-\frac{\partial L}{\partial q_n}=0, \tag{3.7}$$

yields the second-order differential equations (cf. Whittaker 1944, §28)

$$a_{mn}\ddot{q}_m+b_{lmn}\dot{q}_l\dot{q}_m+gq_n=Q_n, \tag{3.8}$$

where

$$b_{lmn}=\tfrac{1}{2}(a_{mn,l}+a_{nl,m}-a_{lm,n}),\quad a_{lm,n}\equiv \partial a_{lm}/\partial q_n. \tag{3.9 a, b}$$

Dissipation may be incorporated by calculating the corresponding generalized forces from the dissipation rate. We assume that the latter may be approximated by $2\rho SF$, where

$$F=\tfrac{1}{2}f_n\dot{q}_n\dot{q}_n \tag{3.10}$$

is Rayleigh's dissipation function (Whittaker 1944, §93) after factoring ρS and neglecting cross-products ($m \neq n$),† and $f_n=g\mathscr{L}_n/\pi\omega_n$, where $\mathscr{L}_n$ is the logarithmic decrement of the nth normal mode. The negative of the corresponding generalized force is $\partial F/\partial \dot{q}_n=f_n\dot{q}_n$, which may be added to the left-hand side of (3.8). See Miles (1967) for the calculation of $\mathscr{L}_n$.

4. Hamiltonian formulation

The generalized momenta and the corresponding Hamiltonian derived from (2.11), (3.1) and (3.4) after factoring out ρS are

$$p_n\equiv(\rho S)^{-1}(\partial T/\partial \dot{q}_n)=a_{mn}\dot{q}_m=d_{nm}\phi_m \tag{4.1 a}$$

or

$$\mathbf{p}=\boldsymbol{a}\dot{\mathbf{q}}=\boldsymbol{d}\boldsymbol{\phi} \tag{4.1 b}$$

and

$$H\equiv(\rho S)^{-1}(T+V)=\tfrac{1}{2}h_{mn}p_mp_n+\tfrac{1}{2}gq_nq_n-Q_nq_n, \tag{4.2}$$

where

$$\boldsymbol{h}\equiv[h_{mn}]=\boldsymbol{d}'^{-1}\boldsymbol{k}\boldsymbol{d}^{-1}=\boldsymbol{a}^{-1} \tag{4.3}$$

is a symmetric matrix with the dimensions of wavenumber. Substituting (3.3) into (4.3) yields

$$h_{mn}=\delta_{mn}k_m+h_{lmn}q_l+\tfrac{1}{2}h_{jlmn}q_jq_l+\dots, \tag{4.4 a}$$

where

$$h_{lmn}=D_{lmn}-C_{lmn}k_mk_n \tag{4.4 b}$$

and

$$h_{jlmn}=D_{jlmn}(k_m+k_n)-2C_{jmi}D_{lni}k_m-2C_{lni}D_{jmi}k_n+2C_{jmi}C_{lni}k_ik_mk_n. \tag{4.4 c}$$

Substituting (4.2) into Hamilton's equations,

$$\dot{p}_n=-\partial H/\partial q_n,\quad \dot{q}_n=\partial H/\partial p_n, \tag{4.5 a, b}$$

yields

$$\dot{p}_n+gq_n+\tfrac{1}{2}h_{lm,n}p_lp_m=Q_n,\quad \dot{q}_n=h_{mn}p_m. \tag{4.6 a, b}$$

† Neglecting cross-products, and hence modal coupling, in the dissipation function is justified in the present context if $\mathscr{L}_n=O(q_n/d)\equiv O(\epsilon)$ as $\epsilon\downarrow 0$. In particular, the modal-coupling terms that would appear in the equations of motion if terms like $f_{mn}\dot{q}_m\dot{q}_n$ ($m\neq n$) were included in F would be eliminated by the averaging operations in §§6 and 7.

5. Perturbation pressure

The perturbation pressure, including that induced by the acceleration of the basin, is given by

$$p(\mathbf{x}, y, t) = -\rho(\dot{\mathbf{u}} \cdot \mathbf{x} + \mathscr{g}y + \phi_t + \tfrac{1}{2}\nabla\phi \cdot \nabla\phi) \tag{5.1}$$

and comprises a spatially independent component, $-\rho\dot{\phi}_0(t)$, that is not directly determined by the preceding formulation. This component may be determined by averaging the free-surface condition $p = 0$ over $y = \eta$ or, equivalently, from the Lagrangian (Luke 1967; Whitham 1974, §13.2)

$$L_* = \iint dS(\mathbf{x}) \int_{-d}^{\eta(\mathbf{x}, t)} p(\mathbf{x}, y, t)\, dy. \tag{5.2}$$

Substituting (5.1) into (5.2) and invoking the Euler–Lagrange condition $\partial L_*/\partial q_0 = 0$ (note that this is not inconsistent with $q_0 = 0$ in the present context, in which both the kinematical and the dynamical conditions are derivable from L_*) yields

$$\dot{\phi}_0 = Q_0 - (1 - \delta_{0n})\, \mathscr{d}_{0n}\dot{\phi}_n - \tfrac{1}{2}\mathscr{k}_{mn,0}\phi_m\phi_n, \tag{5.3}$$

where $$\mathscr{k}_{mn,0} = S^{-1}\iint (\nabla\chi_m \cdot \nabla\chi_n)_{y=\eta}\, dS \tag{5.4a}$$

$$= \delta_{mn}(\mathscr{k}_n^2 + k_n^2) + [C_{lmn}(\mathscr{k}_m k_n^2 + \mathscr{k}_n k_m^2) + D_{lmn}(\mathscr{k}_m + \mathscr{k}_n)]\, q_l + \dots. \tag{5.4b}$$

Invoking the remaining Euler–Lagrange equations (note that L_* does not depend directly on $\dot{\mathbf{q}}$),

$$\frac{\partial L_*}{\partial q_n} = 0, \quad \frac{d}{dt}\frac{\partial L_*}{\partial \dot{\phi}_n} - \frac{\partial L_*}{\partial \phi_n} = 0, \tag{5.5a, b}$$

yields $$\mathscr{d}_{nm}\dot{\phi}_m + \mathscr{g}q_n + \tfrac{1}{2}\mathscr{k}_{lm,n}\phi_l\phi_m = Q_n, \quad \mathscr{d}_{mn}\dot{q}_m = \mathscr{k}_{mn}\phi_m, \tag{5.6a, b}$$

where $\mathscr{d}_{mn}$, $\mathscr{k}_{mn}$ and Q_n are given by (2.9) and (3.5). We note that (5.6*b*) is equivalent to (2.10) and that (5.6*a*, *b*) bear a strong structural similarity to (4.6*a*, *b*), to which they are related by the transformation $\mathbf{p} = \boldsymbol{d}\boldsymbol{\phi}$. Eliminating $\{\phi_n\}$ between (5.6*a*, *b*) yields (3.8) after a non-trivial reduction.

The first-order differential equations (5.6) have the virtue of not requiring the prior inversion of the matrix $\boldsymbol{k}$, although this inversion is implicit in their solution; moreover, L_* provides $\dot{\phi}_0$ without the necessity of a separate, *ad hoc* argument. On balance, however, it appears that either the Lagrangian formulation of §3 or the Hamiltonian formulation of §4 offers significant advantages *vis-à-vis* the formulation provided by L_* in the present context.

6. Weakly coupled free oscillations

The simplest problem governed by (3.6) is that of weakly coupled free oscillations, for which $\dot{\mathbf{u}} = \dot{v} = 0$ ($\mathscr{g} = g$, $Q_n = 0$). The linear approximation yields uncoupled oscillations at the natural frequencies ($\omega_n^2 \equiv g/\mathscr{a}_n$), which then provide the basis for a perturbation solution.

Let $q_1 = A_1 \cos \omega t$ and $\omega = \omega_1$ represent the first approximation to the solution of (3.8) for any particular mode ($n = 1$ does not necessarily imply the dominant mode) with $A_1 = O(\epsilon d), \epsilon \ll 1$.† The quadratic (in q_1) terms in the differential equation then excite time-independent and second-harmonic components; accordingly, we consider a second approximation of the form

$$q_n = \delta_{1n} A_1 \cos \omega t + A_{n0} + A_{n2} \cos 2\omega t, \tag{6.1}$$

in which A_{n0} and A_{n2} are $O(\epsilon^2 d)$. Substituting (6.1) into (3.6), setting $g = g$ and $Q_n = 0$, invoking (3.3a) for a_{mn}, neglecting terms that are $O(\epsilon^6)$, and averaging over the period $2\pi/\omega$, we obtain

$$\begin{aligned}\langle L\rangle = \tfrac{1}{2}\omega^2\{\tfrac{1}{2}a_1 A_1^2 + 2a_m A_{m2}^2 + \tfrac{1}{2}a_{m11} A_1^2 A_{m0} + (a_{11m} - \tfrac{1}{4}a_{m11}) A_1^2 A_{m2} \\ + \tfrac{1}{16}a_{1111} A_1^4\} - \tfrac{1}{4}g(A_1^2 + 2A_{m0}A_{m0} + A_{m2}A_{m2}).\end{aligned} \tag{6.2}$$

Requiring $\langle L\rangle$ to be stationary with respect to each of A_{n0}, A_{n2} and A_1 (which is equivalent to the invocation of Hamilton's principle for the assumed motion) yields

$$A_{n0} = \frac{1}{4}\left(\frac{a_{n11}}{a_1}\right) A_1^2, \quad A_{n2} = -\frac{1}{4}\left(\frac{4a_{11n} - a_{n11}}{4a_n - a_1}\right) A_1^2, \tag{6.3a, b}$$

in which the first approximation $\omega^2 = g/a_1$ has been invoked, and

$$[\omega^2\{a_1 + a_{m11} A_{m0} + (2a_{11m} - \tfrac{1}{2}a_{m11}) A_{m2} + \tfrac{1}{4}a_{1111} A_1^2\} - g] A_1 = 0. \tag{6.4}$$

Substituting (6.3) into (6.4) and solving for ω^2 on the assumption that $A_1 \neq 0$ yields

$$(\omega/\omega_1)^2 = 1 + \tfrac{1}{4}(A_1/a_1)^2 [\tfrac{1}{2}a_1(4a_m - a_1)^{-1} (4a_{11m} - a_{m11})^2 - a_{m11}a_{m11} - a_1 a_{1111}]. \tag{6.5}$$

The determination of the coefficients a_{11m}, a_{m11} and a_{1111} in (6.3) and (6.5) may require extensive, albeit straightforward, calculation for a particular configuration; nevertheless, the terseness of the preceding solution is rather striking in comparison with the usual perturbation solutions (cf. Tadjbakhsh & Keller 1960; Verma & Keller 1962; Mack 1962).

Substituting (6.1) into (2.11), invoking (2.14) for ℓ_{mn}, and then calculating ϕ_0 from (5.3) we obtain (n not summed)

$$\phi_n = -\delta_{1n}\omega A_1 \sin \omega t + \omega a_n(\tfrac{1}{2}D_{11n} a_1 A_1^2 - 2A_{n2}) \sin 2\omega t \tag{6.6}$$

and
$$\phi_0 = \tfrac{1}{4}\omega^2 A_1^2\{-(1 - T^{-2}) + (3 + T^{-2}) \cos 2\omega t\} \quad (T \equiv \tanh k_1 d). \tag{6.7}$$

Carrying out the calculation for the dominant two-dimensional mode in a basin of length π/k [see (2.19)] and replacing A_1 by $A/\surd 2$ yields (the second-order term is identically zero for $n = 1$ since $C_{111} = 0$)

$$\eta = A \cos \omega t \cos kx + \tfrac{1}{8}kA^2[T^{-1} + T + T^{-1}(3T^{-2} - 1) \cos 2\omega t] \cos 2kx, \tag{6.8}$$

$$\begin{aligned}\phi = \tfrac{1}{8}\omega A^2[-S^{-2}\omega t + (2 - \tfrac{1}{2}S^{-2}) \sin^2 2\omega t] - (\omega/k) AS^{-1} \sin \omega t \cos kx \cosh k(y+d) \\ - \tfrac{3}{16}\omega^2 S^{-4} \sin 2\omega t \cos 2kx \cosh 2k(y+d) \quad (S \equiv \sinh kd),\end{aligned} \tag{6.9}$$

and
$$(\omega/\omega_1)^2 = 1 + \tfrac{1}{32}k^2A^2(9T^{-4} - 12T^{-2} - 3 - 2T^2). \tag{6.10}$$

† Choosing ϵ such that $A_1 \equiv \epsilon d$ yields a scaling equivalent to that of § 7; however, the results in this section are independent of the choice of ϵ, which serves only as an order-of-magnitude parameter.

Setting $T = 1$ in (6.10) yields Rayleigh's (1915) result for deep-water waves. Comparing (6.8) and (6.9) with the results cited in equation (27.63) of the *Encyclopedia of Physics* (Wehausen & Laitone 1960) indicates that the signs of the

$$\cos 2\sigma t \cos 2mx \quad \text{and} \quad \sin 2\sigma t \cos 2mx$$

terms therein ($\sigma \equiv \omega$, $m \equiv k$) should be reversed [Professor Wehausen (private communication) agrees]. A similar comparison with Tadjbakhsh & Keller (1960) indicates that $\omega_0^2 - \omega_0^{-2} \equiv T - T^{-1}$ in their equation (30) should read $\omega_0^2 + \omega_0^{-2}$. It appears that this error may have affected their third approximation, since the deep-water limits ($k_1 d \uparrow \infty$) of b_{13} and b_{33} in their equation (38), $\frac{9}{16}$ and $-\frac{3}{16}$ respectively, disagree with the corresponding values, $\frac{1}{2}$ and $\frac{1}{2}$, implied by Rayleigh's (1915) solution; however, their result for ω agrees with (6.10).

The calculation of $(\omega/\omega_1)^2$ from (6.5) for the rectangular mode considered by Verma & Keller agrees with the result given by their equations (36) and (39).

7. Resonantly coupled free oscillations ($\omega_2 \doteqdot 2\omega_1$)

The perturbation solution of §6 fails (in particular, $|A_{n2}| \gg \epsilon^2 d$) if $\omega_n \equiv \omega_2$ approximates $2\omega_1$ (there is no loss of generality in choosing the subscripts 1 and 2), and q_1 and q_2 then are *resonantly coupled*.† Harmonic motions are still possible but only for special initial conditions, and in general the two modes must be expected to have slowly varying amplitudes and phases.

Of the several asymptotic techniques that are available for attacking this problem, the most efficient (at least for the derivation of integral invariants of the motion) appears to be the introduction of the *action* and *angle* variables $\mathscr{p}_n$ and $\mathscr{q}_n$ through the canonical transformation (cf. Whittaker 1944, §193; Mettler 1963)

$$q_n = (2\omega_n \mathscr{p}_n/g)^{\frac{1}{2}} \cos \mathscr{q}_n, \quad p_n = -(2g\mathscr{p}_n/\omega_n)^{\frac{1}{2}} \sin \mathscr{q}_n, \tag{7.1 a, b}$$

under which (4.5) goes over to

$$\dot{\mathscr{p}}_n = -\partial H/\partial \mathscr{q}_n, \quad \dot{\mathscr{q}}_n = \partial H/\partial \mathscr{p}_n. \tag{7.2 a, b}$$

Substituting (7.1) into (4.2) with $\mathscr{g} = g$ and $Q_n = 0$ and retaining only the first two terms in the expansion (4.4a) yields

$$H = \omega_n \mathscr{p}_n + h_{lmn}(2g\omega_l/\omega_m\omega_n)^{\frac{1}{2}} (\mathscr{p}_l \mathscr{p}_m \mathscr{p}_n)^{\frac{1}{2}} \cos \mathscr{q}_l \sin \mathscr{q}_m \sin \mathscr{q}_n + O(\epsilon^4). \tag{7.3}$$

The first approximation, obtained by substituting $H = \omega_n \mathscr{p}_n$ into (7.2), is described by $\dot{\mathscr{p}}_n = 0$ and $\dot{\mathscr{q}}_n = \omega_n$. The next approximation contains terms [after expressing the trigonometric products in (7.3) in terms of cos $(\mathscr{q}_l \pm \mathscr{q}_m \pm \mathscr{q}_n)$] that oscillate with frequencies of approximately $|\omega_l \pm \omega_m \pm \omega_n|$ (with all four sign combinations), of which (by assumption) only $\omega_2 - 2\omega_1$ is small compared with ω_1. The slow variations of $\mathscr{p}_n$ and $\mathscr{q}_n$ therefore are significant only for $n = 1, 2$ and may be calculated by neglecting those trigonometric terms with arguments other than $2\mathscr{q}_1 - \mathscr{q}_2$ or, equivalently, by introducing fast and slow times and averaging

† This is a special case of the second-order resonant interaction defined by

$$\omega_1 - \omega_2 + \omega_3 \doteqdot 0.$$

over the former to obtain (where, here and subsequently, n is summed over 1 and 2)

$$\langle H\rangle = \omega_n \mathscr{p}_n + \mathscr{C}\mathscr{p}_1\mathscr{p}_2^{\frac{1}{2}}\cos\beta, \quad \beta = 2\varphi_1 - \varphi_2, \tag{7.4a, b}$$

where

$$\mathscr{C} = (g/2\omega_2)^{\frac{1}{2}}\{h_{112} - (\omega_2/2\omega_1)\,h_{211}\} \doteqdot \tfrac{1}{2}(g/\omega_1)^{\frac{1}{2}}(h_{112} - h_{211}) \tag{7.5}$$

is a nonlinear–coupling parameter (in which $h_{121} = h_{112}$ has been invoked.) A more detailed justification of the argument follows from the consideration that Hamilton's action integral, $\int L\,dt$, is dominated by the slowly oscillating terms, which contain the factor $1/(\omega_2 - 2\omega_1)$. The resulting error factor in the end results as $\epsilon \downarrow 0$ with (by hypothesis) q_1/d, q_2/d and $\beta = O(\epsilon)$ is $1 + O(\epsilon)$ uniformly with respect to t.

Substituting the approximation (7.4*a*) into (7.2) yields

$$\dot{\mathscr{p}}_1 = 2\mathscr{C}\mathscr{p}_1\mathscr{p}_2^{\frac{1}{2}}\sin\beta, \quad \dot{\mathscr{p}}_2 = -\mathscr{C}\mathscr{p}_1\mathscr{p}_2^{\frac{1}{2}}\sin\beta, \tag{7.6a, b}$$

$$\dot{\varphi}_1 = \omega_1 + \mathscr{C}\mathscr{p}_2^{\frac{1}{2}}\cos\beta, \quad \dot{\varphi}_2 = \omega_2 + \tfrac{1}{2}\mathscr{C}\mathscr{p}_1\mathscr{p}_2^{-\frac{1}{2}}\cos\beta. \tag{7.6c, d}$$

It follows from (7.6*a*, *b*) that the action integral (or *adiabatic invariant*)

$$\mathscr{p}_1 + 2\mathscr{p}_2 \equiv \mathscr{E} \tag{7.7a}$$

is a constant of the motion.† Moreover, it follows from the conservation of energy that H, and hence

$$\langle H\rangle - \tfrac{1}{2}\omega_2\mathscr{E} = \mathscr{p}_1(\omega_1 - \tfrac{1}{2}\omega_2 + \mathscr{C}\mathscr{p}_2^{\frac{1}{2}}\cos\beta) \equiv \mathscr{H}, \tag{7.7b}$$

also is a constant of the motion. The availability of these two integrals permits the integration of (7.6) in terms of elliptic functions. Before proceeding further, however, we find it expedient to introduce the dimensionless, slowly varying amplitudes and phases $\mathscr{A}_n$ and α_n, such that

$$q_n(t) = \epsilon d\mathscr{A}_n(\tau)\cos\{\omega_n t + \alpha_n(\tau)\} \quad (n = 1, 2), \quad \tau = \tfrac{1}{2}c\omega_1 t, \tag{7.8a, b}$$

through the transformation

$$\mathscr{p}_n = \tfrac{1}{2}\epsilon^2 g d^2\omega_n^{-1}\mathscr{A}_n^2(\tau), \quad \varphi_n = \omega_n t + \alpha_n(\tau), \tag{7.9a, b}$$

where

$$c = \tfrac{1}{2}\epsilon(gd/\omega_1^2)(h_{112} - h_{211}) = \tfrac{1}{2}\epsilon C_{112}d\,(k_2^2 - k_1^2 - 3\ell_1^2)/\ell_1 \tag{7.10}$$

is a dimensionless, $O(\epsilon)$ counterpart of $\mathscr{C}$ that has been reduced with the aid of (2.15), (2.18), (4.4*b*) and $\ell_2 \doteqdot 4\ell_1$.

Substituting (7.9) into (7.4*b*), (7.6) and (7.7) and letting $\epsilon \downarrow 0$ yields

$$\dot{\mathscr{A}}_1 = \mathscr{A}_1\mathscr{A}_2\sin\beta, \quad \dot{\mathscr{A}}_2 = -\mathscr{A}_1^2\sin\beta, \tag{7.11a, b}$$

$$\dot{\alpha}_1 = \mathscr{A}_1\cos\beta, \quad \dot{\alpha}_2 = (\mathscr{A}_1^2/\mathscr{A}_2)\cos\beta, \tag{7.11c, d}$$

$$\mathscr{A}_1^2 + \mathscr{A}_2^2 = e \equiv 1, \quad \delta\mathscr{A}_1^2 + \mathscr{A}_1^2\mathscr{A}_2\cos\beta = h, \tag{7.12a, b}$$

and

$$\beta = 2\delta\tau + 2\alpha_1 - \alpha_2, \tag{7.13}$$

where

$$\delta = (2\omega_1 - \omega_2)/c\omega_1 \tag{7.14}$$

is an $O(1)$ measure of the modal separation, e and h are constants of the motion that are determined by the initial conditions (there is no relation between h and

† Whittaker (1944, §195 ff.) uses the term "adelphic [from αδελφικός or *brotherly*] integral" for the corresponding invariant of H.

$\hbar_{mn}$), and the dots now imply differentiation with respect to τ. Choosing ϵ such that

$$T+V = \tfrac{1}{2}\rho S g d^2 \epsilon^2 (\mathscr{A}_1^2+\mathscr{A}_2^2) \equiv \tfrac{1}{2}\rho S g d^2 \epsilon^2 \tag{7.15}$$

in the limit $\epsilon \downarrow 0$ yields $\varepsilon = 1$.

We remark that (7.11) also may be obtained by substituting (7.8) into (3.6) with $\mathcal{g} = g$ and $Q_n = 0$, averaging over t with τ fixed to obtain

$$\langle L\rangle = \tfrac{1}{4}c\epsilon^2 g d^2 (\mathscr{A}_1^2\dot{\alpha}_1 + \tfrac{1}{2}\mathscr{A}_2^2\dot{\alpha}_2 - \mathscr{A}_1^2\mathscr{A}_2\cos\beta)\{1+O(\epsilon)\}, \tag{7.16}$$

and regarding $\mathscr{A}_1$, $\mathscr{A}_2$, α_1 and α_2 as generalized co-ordinates. This procedure is somewhat more direct in the present context, but it does not lead as naturally to the invariant (7.12*b*) [the derivation of (7.12*a*) from (7.11) is rather obvious, but that of (7.12*b*) is less so].†

The range of $\hbar$ in (7.12*b*) is limited by the constraint (7.12*a*) and the requirement that $\mathscr{A}_1$ and $\mathscr{A}_2$ be real ($0 < \mathscr{A}_1^2, \mathscr{A}_2^2 < 1$). The extrema are given by (see figure 1)

$$\mathscr{A}_1^2 = 1-\mathscr{A}^2, \quad \mathscr{A}_2\cos\beta = \mathscr{A}, \quad \sin\beta = 0 \tag{7.17a, b, c}$$

and
$$\hbar = (\delta+\mathscr{A})(1-\mathscr{A}^2) \equiv \hbar_{\pm}(\delta) = -\hbar_{\mp}(-\delta), \tag{7.17d}$$

where
$$\mathscr{A} = \tfrac{1}{3}\{-\delta \pm (\delta^2+3)^{\frac{1}{2}}\} \equiv \mathscr{A}_{\pm}, \tag{7.17e}$$

the signs in (7.17*d, e*) are vertically ordered, and $\mathscr{A}_{\pm}$ is excluded for $\mp\,\delta > 1$. The admissible ranges then are

$$\hbar_- < \hbar < 0 \quad (\delta < -1), \tag{7.18a}$$

$$\hbar_- < \hbar < \hbar_+ \quad (-1 < \delta < 1) \tag{7.18b}$$

and
$$0 < \hbar < \hbar_+ \quad (\delta > 1). \tag{7.18c}$$

It follows from (7.11) and (7.17) that the extrema $\hbar = \hbar_{\pm}(\delta)$ render $\mathscr{A}_n$ and $\dot{\alpha}_n$ constant and therefore correspond to harmonic motions. The equivalent frequency for q_n [see (7.8)] is

$$\omega_n + \tfrac{1}{2}\epsilon\omega_1\dot{\alpha}_n = n\omega_1(1+\tfrac{1}{2}\epsilon\mathscr{A}_{\pm}) \equiv n\omega_{\pm} \quad (n = 1, 2). \tag{7.19}$$

It follows from (7.14) and (7.19) that

$$\omega_- < \omega_1 < (\omega_+) < \tfrac{1}{2}\omega_2 \quad (\delta < -1), \tag{7.20a}$$

$$\omega_- < \omega_1 < \tfrac{1}{2}\omega_2 < \omega_+ \quad (-1 < \delta < 0), \tag{7.20b}$$

$$\omega_- < \tfrac{1}{2}\omega_2 < \omega_1 < \omega_+ \quad (0 < \delta < 1), \tag{7.20c}$$

$$\tfrac{1}{2}\omega_2 < (\omega_-) < \omega_1 < \omega_+ \quad (\delta > 1), \tag{7.20d}$$

where the parentheses around $\omega_{\pm}$ in (7.20*a, d*) imply the aforementioned exclusions; accordingly, harmonic oscillations with frequencies between ω_1 and $\frac{1}{2}\omega_2$ are impossible for $|\delta| > 1$. A straightforward stability analysis reveals that the admissible harmonic motions described by (7.17) and (7.19) are stable with respect to small perturbations.

† Professor Whitham has pointed out (private communication) that (7.12*a*) and (7.12*b*) can be derived from (7.16) through two independent applications of Noether's theorem.

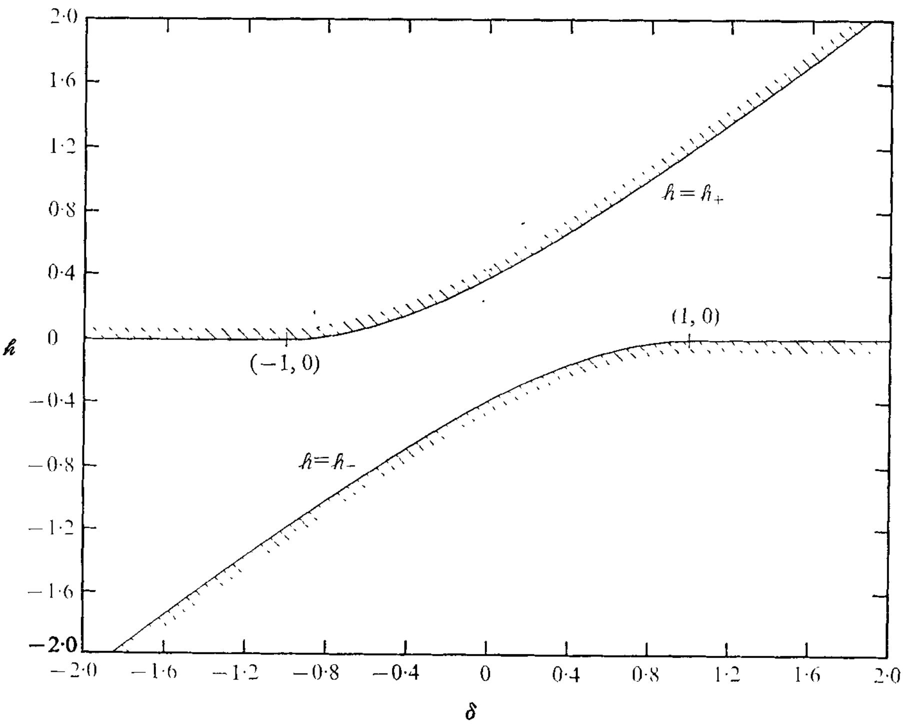


FIGURE 1. The extrema for $h(\delta)$, as given by (7.17) and (7.18).

The anharmonic solutions of (7.11) are conveniently represented by their trajectories in an x, $\dot{x}$ phase plane (phase is now used in the conventional sense of nonlinear differential equations), where

$$x = \mathscr{A}_1^2 = 1 - \mathscr{A}_2^2 \tag{7.21}$$

is a dimensionless counterpart of the action variable p_1. Eliminating $\cos\beta$ from (7.11) with the aid of (7.12*b*) then yields

$$\tfrac{1}{4}\dot{x}^2 = x^2(1-x) - (\delta x - h)^2 \equiv f(x) \quad (0 \leqslant x \leqslant 1) \tag{7.22}$$

and

$$\dot{\alpha}_1 = -\delta + hx^{-1}, \quad \dot{\alpha}^2 = (h - \delta x)/(1-x). \tag{7.23a,b}$$

It suffices to consider $h \geqslant 0$, since $f(x)$ remains unchanged if the signs of both h and δ are changed.

The cubic $f(x)$ has two positive zeros, x_1 and x_2, where $0 < x_1 < x_2 < 1$, if h is in one of the admissible ranges (7.18); see figure 2. These zeros coincide if either $h = h_-$ or $h = h_+$ and then are stable singular points (*centres*) in the phase plane (but $h = h_\pm$ is excluded for $\mp\delta > 1$). Representative phase-plane trajectories for $0 < h < h_+$ are plotted in figure 3. We observe that the trajectories for $h \geqslant 0$ are nested for $\delta \leqslant 0$ but not for $\delta > 0$.

The period T, defined as the time for x to go from x_1 to x_2 and back to x_1, is given by

$$\tfrac{1}{2}c\omega_1 T = \int_{x_1}^{x_2} \{f(x)\}^{-\frac{1}{2}}\,dx = 2(x_2 - x_3)^{-\frac{1}{2}} K\{(x_2 - x_1)^{\frac{1}{2}}/(x_2 - x_3)^{\frac{1}{2}}\} \equiv \hat{T}, \tag{7.24}$$

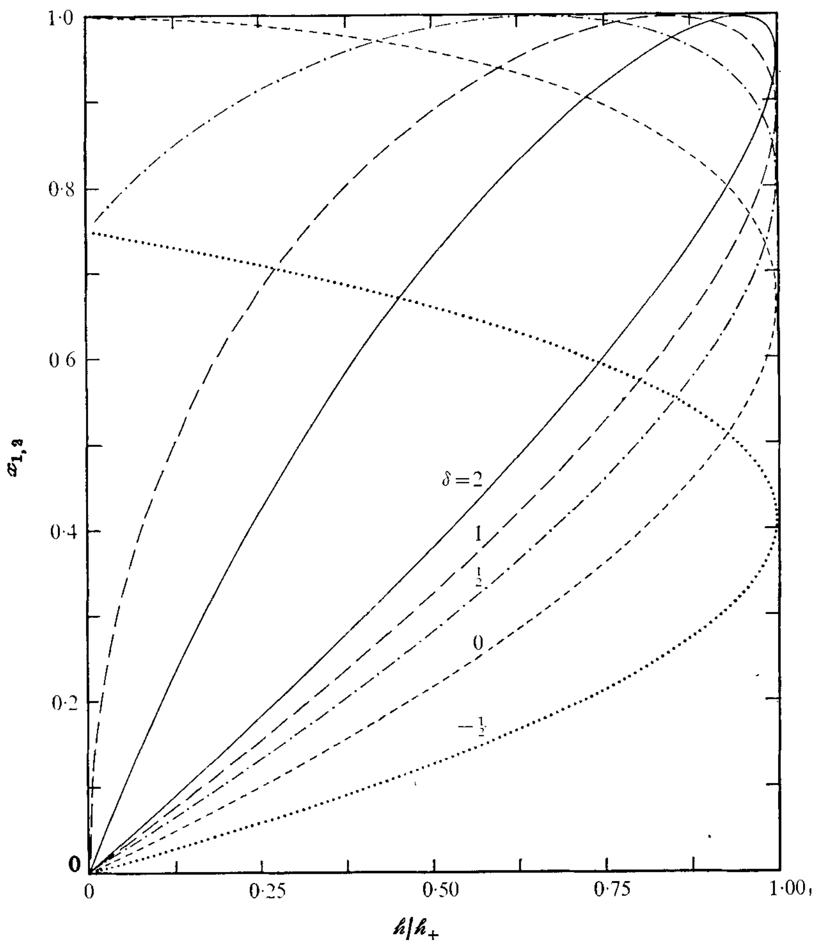


FIGURE 2. The real zeros of $f(x)$, $0 < x_1 < x_2 < 1$; see (7.22).

where $x_3 = -\hbar^2/x_1 x_2$ is the negative zero of f, and K is an elliptic integral of the first kind. This dimensionless period is plotted in figure 4. We note the finite limit

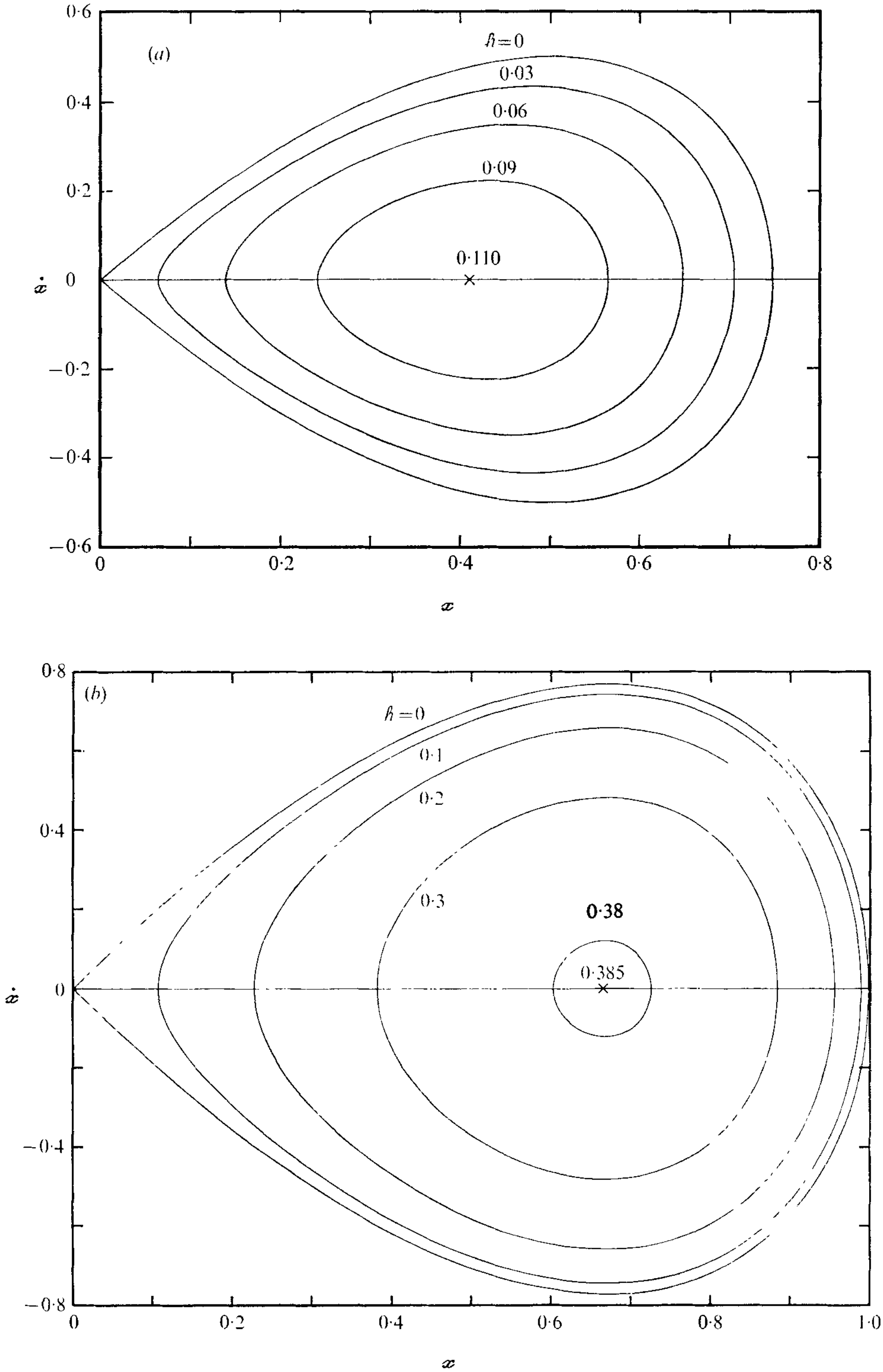
$$\hat{T} \to \pi(1+\delta^2+2\delta\mathscr{A}_+)^{-\frac{1}{2}} \quad (\hbar \uparrow \hbar_+). \tag{7.25}$$

The corresponding amplitude vanishes like $(\hbar_+ - \hbar)^{\frac{1}{2}}$.

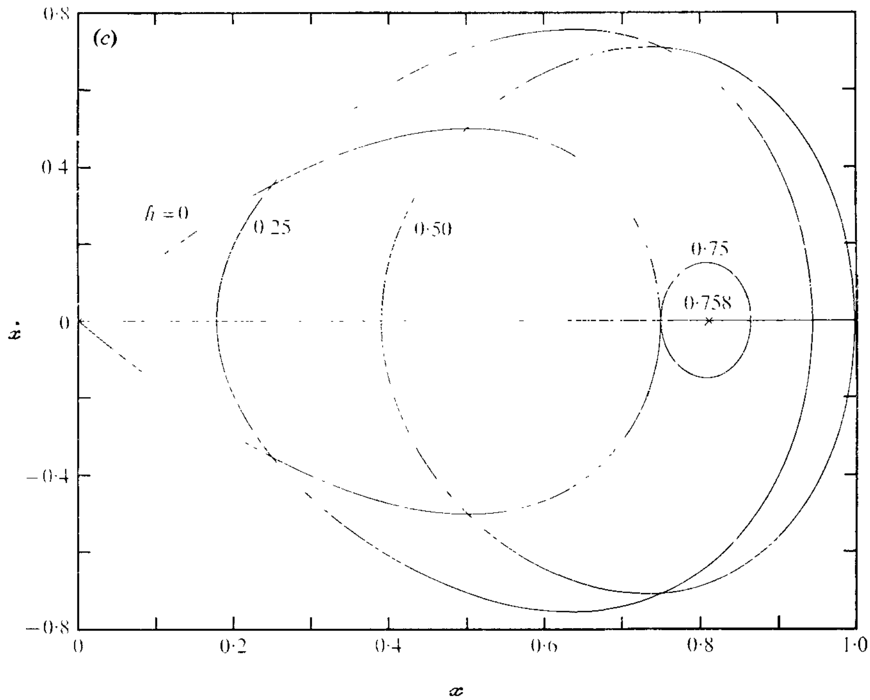
The integration of (7.22) for $\hbar = 0$ and $\delta^2 < 1$, for which $f(x)$ has a double zero at $x = 0$, which then is a *nodal point*, yields

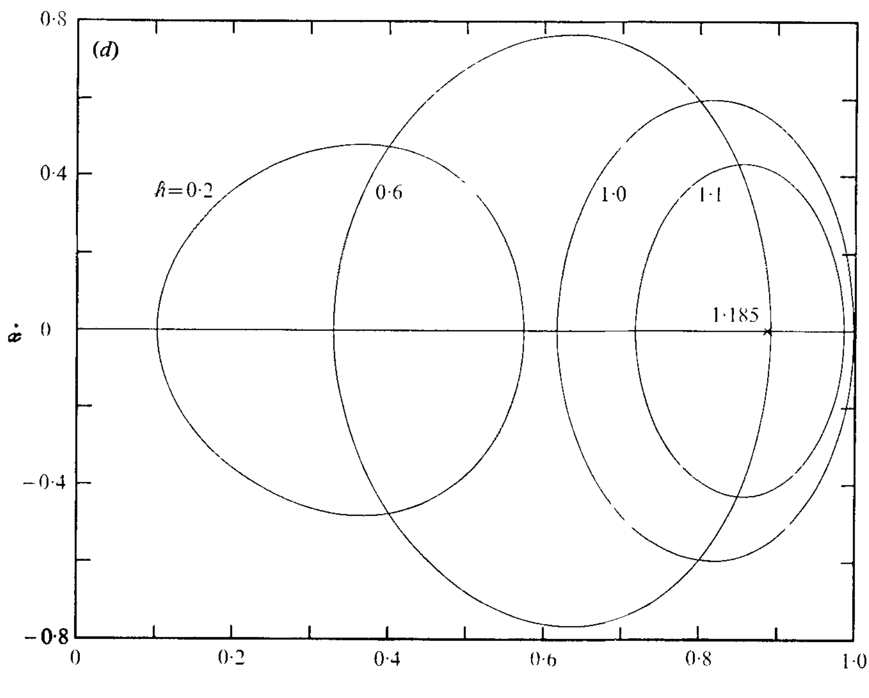
$$x = (1-\delta^2)\,\mathrm{sech}^2\{(1-\delta^2)^{\frac{1}{2}}(\tau-\tau_0)\} \quad (\hbar = 0, \quad \delta^2 < 1), \tag{7.26}$$

where τ_0 is a constant of integration. It follows that, for those special initial conditions corresponding to $\hbar = 0$, q_1 has constant frequency and (for $\tau > \tau_0$) monotonically decaying amplitude, such that all of the energy is ultimately transferred to the second mode, which then continues to oscillate at its natural frequency ω_2 (since $\dot{\alpha}_2 \sim 0$ as $\tau \to \infty$). This solution is especially striking for $\delta = 0$ and the initial conditions $\mathscr{A}_1 = 1$ and $\mathscr{A}_2 = 0$ at $\tau = \tau_0$: the total energy then



FIGURES 3(*a*, *b*). For legend see page 434.





FIGURES 3 (c, d). For legend see next page.

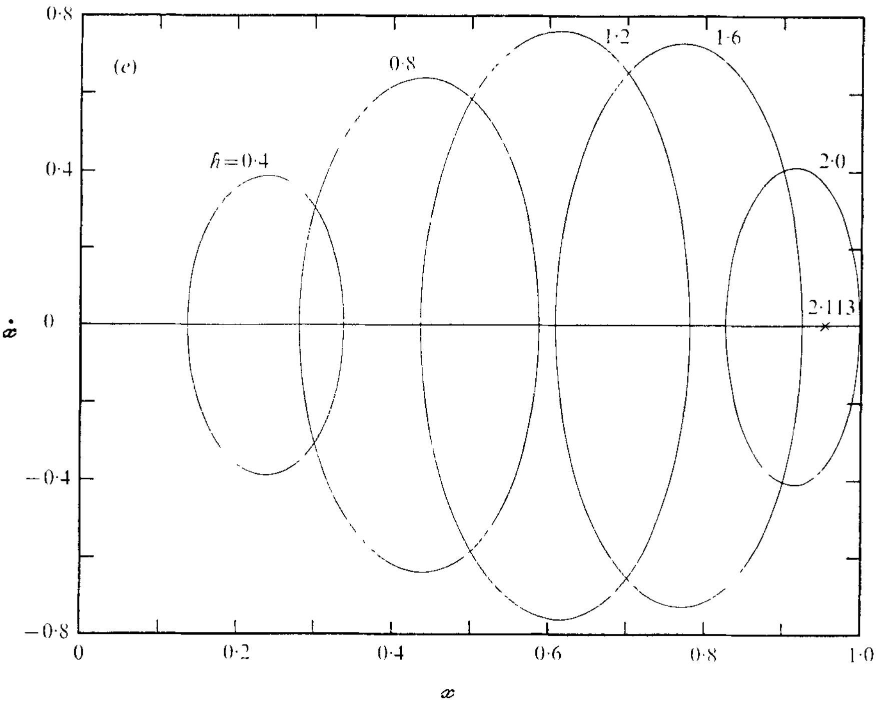


FIGURE 3. The phase-plane trajectories calculated from (7.22). (*a*) $\delta = -\frac{1}{2}$, for which the singular point corresponds to $\hbar = \hbar_+ = 0{\cdot}110$. (*b*) $\delta = 0$, for which the singular point corresponds to $\hbar = \hbar_+ = 0{\cdot}385$. (*c*) $\delta = \frac{1}{2}$, for which the singular point corresponds to $\hbar = \hbar_+ = 0{\cdot}758$. (*d*) $\delta = 1$, for which the singular point corresponds to $\hbar = \hbar_+ = 1{\cdot}185$. (*e*) $\delta = 2$, for which the singular point corresponds to $\hbar = \hbar_+ = 2{\cdot}113$.

appears initially in the first mode and is ultimately transferred entirely to the second mode.

The integration of (7.22) for all $\hbar$ other than $\hbar = \hbar_\pm$ or $\hbar = 0$ leads to elliptic functions and is considered in some detail by Struble & Heinbockel (1963), whose equations (14) and (15) are equivalent to our (7.11) and (7.12*a*) after the transformation

$$\mathscr{A}_1 = \delta^{-1}B, \quad \mathscr{A}_2 = (\beta\delta)^{-1}(\beta+\tfrac{1}{2})^{\frac{1}{2}}A, \quad \delta = (\beta\delta\epsilon)^{-1}(2\beta-1)(\beta+\tfrac{1}{2})^{\frac{1}{2}}, \quad (7.27\,a,b,c)$$

with β, δ and ϵ on the right-hand sides of (7.27) defined as in their paper; however, they do not identify the invariant (7.12*b*) and, perhaps for this reason, do not obtain the explicit solution (7.26). This special case $\delta = 0$ is considered for capillary waves by McGoldrick (1970), who, following Simmons (1969), does identify both invariants. The case $\hbar = \delta$, for which $\dot{\alpha}_2 = \delta$ (so that q_2 has constant frequency), also appears to merit special mention.

The specific application of the results in this section depends essentially on the parameters ϵ and δ, (7.10) and (7.14). It is readily verified that $C_{112} = 0$ for a rectangular basin except for the wavenumber pairs: (*a*) $k_{2x} = 2k_{1x}$, $k_{2z} = 2k_{1z}$; (*b*) $k_{2x} = 2k_{1x} \neq 0$, $k_{1z} \neq 0$, $k_{2z} = 0$; (*c*) $k_{2z} = 2k_{1z} \neq 0$, $k_{1x} \neq 0$, $k_{2x} = 0$. It can be demonstrated that $w_2 = 2\omega_1$ is impossible for (*b*) and (*c*) and possible for (*a*) only

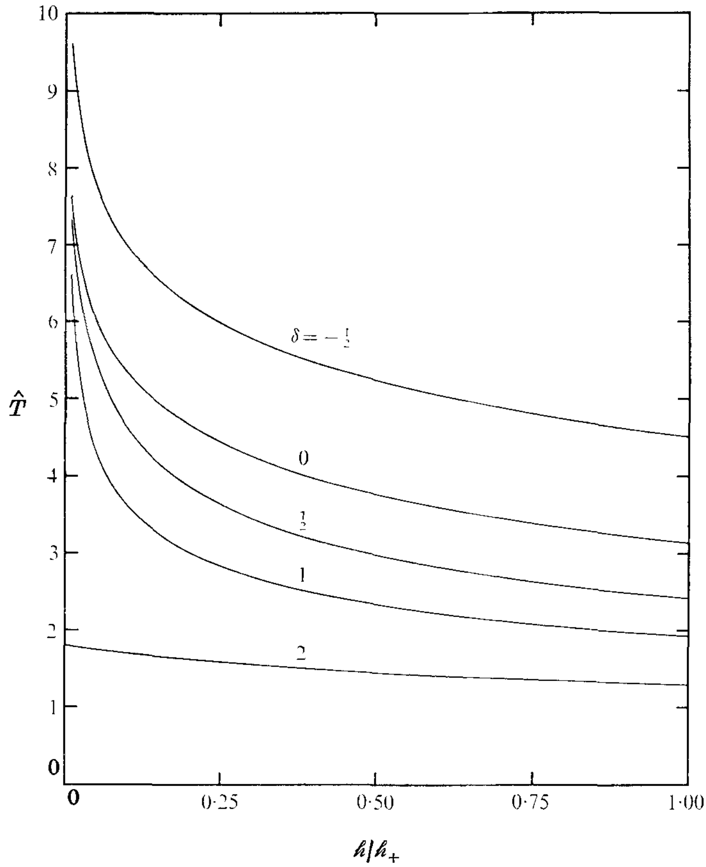


FIGURE 4. The dimensionless period for the slow modulation, as given by (7.24).

in the shallow-water limit $k_1 d \downarrow 0$. The results of this section are typically inadequate in this limit, however, since more than two modes may be resonantly coupled. In particular

$$\omega_n = nk(gd)^{\frac{1}{2}}\left(1 - \tfrac{1}{6}n^2k^2d^2 + \ldots\right) \quad (nkd \downarrow 0) \tag{7.28}$$

for two-dimensional waves in a tank of length π/k, and all modes for which $nkd \ll 1$ are resonantly coupled (cf. Bryant 1973).

Mack (1962) obtains results equivalent to (7.17) and (7.19) for resonant interactions between the first and third (fourth) axisymmetric modes in a circular cylinder of radius a and $h/a \doteqdot 0{\cdot}198$ $(0{\cdot}347)$. A straightforward calculation (numerical values of the Bessel-function correlation integrals are given by McIntyre 1972) yields $\varepsilon/\epsilon = -0{\cdot}043$ $(-0{\cdot}011)$ for the corresponding coupling parameter(s).† Mack evidently overlooks the necessity for special initial conditions for harmonic response and does not consider anharmonic motion.

† These values are an order of magnitude smaller than the value, $\varepsilon/\epsilon \doteqdot 0{\cdot}48$, for the double pendulum sketched in figure 3 of Rott's (1970) paper.

8. Forced oscillations ($\omega \doteqdot \omega_1 \doteqdot \frac{1}{2}\omega_2$)

We now suppose that the basin is oscillating in the x direction with displacement amplitude X and frequency $\omega \doteqdot \omega_1 \doteqdot \frac{1}{2}\omega_2$. The generalized force calculated from (3.5a, b) then is

$$Q_n = \omega^2 X x_n \cos \omega t \quad (n = 1, 2). \tag{8.1}$$

The response in the limit $\epsilon \downarrow 0$ may be posed in the form (7.8), but it proves more convenient (at least for numerical integration) to introduce the alternative representation

$$q_n(t) = \epsilon d\{C_n(\tau) \cos n\omega t + S_n(\tau) \sin n\omega t\}, \quad \tau = \tfrac{1}{2}c\omega t. \tag{8.2a, b}$$

Substituting (8.1) and (8.2) into (3.6) with $\mathscr{g} = g$, neglecting terms that are $O(\epsilon^4)$, and averaging over t with τ fixed yields

$$\begin{aligned}\langle L\rangle = \tfrac{1}{4}c\epsilon^2 d^2 g\{n^{-1}(S_n \dot{C}_n - C_n \dot{S}_n) - \nu_n(C_n^2 + S_n^2) \\ - (C_1^2 - S_1^2)\, C_2 - 2S_1 C_1 S_2\} + \tfrac{1}{2}\epsilon d\omega^2 X x_1 C_1,\end{aligned} \tag{8.3}$$

where the dots imply differentiation with respect to τ, n is summed over 1, 2, and

$$\nu_n = (\omega_n^2 - n^2\omega^2)/c\omega_n^2 \doteqdot 2(\omega_n - n\omega)/c\omega_n \quad (n = 1, 2). \tag{8.4}$$

Note that ω may be approximated by ω_1 and conversely throughout this section except in the numerator of ν_n. Invoking Lagrange's equations for $\langle L\rangle$, with C_1, C_2, S_1, and S_2 as generalized co-ordinates, and incorporating damping (see last paragraph in §3), we obtain

$$\dot{C}_1 + \alpha_1 C_1 - \nu_1 S_1 + S_1 C_2 - S_2 C_1 = 0, \tag{8.5a}$$

$$\dot{S}_1 + \alpha_1 S_1 + \nu_1 C_1 + C_1 C_2 + S_1 S_2 = \mu, \tag{8.5b}$$

$$\dot{C}_2 + 2\alpha_2 C_2 - 2\nu_2 S_2 - 2S_1 C_1 = 0, \tag{8.5c}$$

$$\dot{S}_2 + 2\alpha_2 S_2 + 2\nu_2 C_2 + C_1^2 - S_1^2 = 0, \tag{8.5d}$$

where

$$\alpha_n = \mathscr{L}_n/\pi c, \tag{8.6†}$$

$$\mu = (\epsilon c d g)^{-1}\omega^2 X x_1 \doteqdot 2(\omega_1^2/\epsilon g d)^2 (\hbar_{112} - \hbar_{211})^{-1} X x_1, \tag{8.7}$$

and ϵ is as yet undefined. We proceed on the hypothesis $\hbar_{112} > \hbar_{211}$ and choose ϵ such that $\mu \equiv 1$; the corresponding choice for $\hbar_{112} < \hbar_{211}$ is $\mu \equiv -1$, which would require only that the signs of C_1 and S_1 be changed throughout the remainder of this section.

Harmonic solutions with no damping

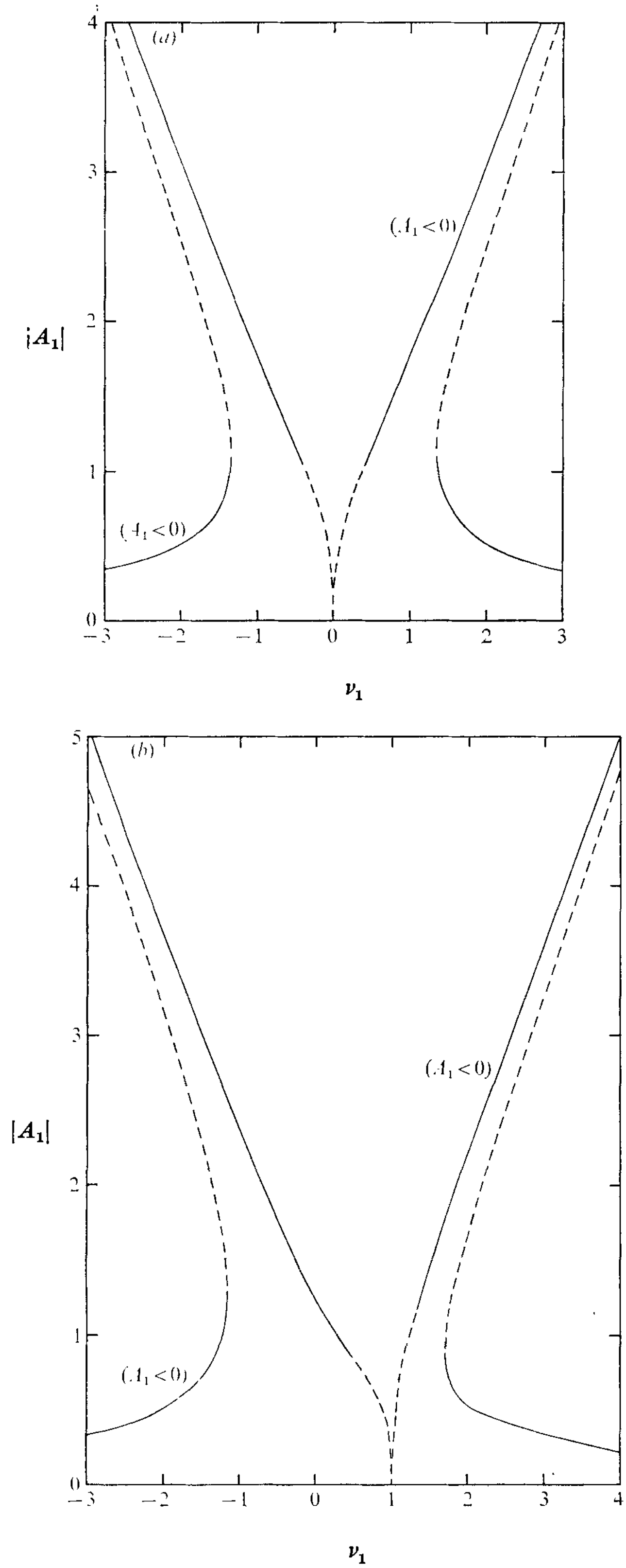
Setting $\alpha_n = S_n = 0$ and $C_n = A_n$ (constant) in (8.5) yields

$$A_1(\nu_1 + A_2) = 1, \quad 2\nu_2 A_2 + A_1^2 = 0. \tag{8.8a, b}$$

Eliminating A_2 between (8.8a, b), introducing δ from (7.14), and solving the resulting equation for ν_1 (rather than solving it as a cubic in A_1) yields

$$\nu_1 = \tfrac{1}{2}A_1^{-1}\{1 + \delta A_1 \pm [(1 - \delta A_1)^2 + 2A_1^4]^{\frac{1}{2}}\}. \tag{8.9}$$

† The damping parameter α_n is unrelated to the phase constant α_n in §7, which does not appear in this section.



FIGURES 5 (a, b). For legend see next page.

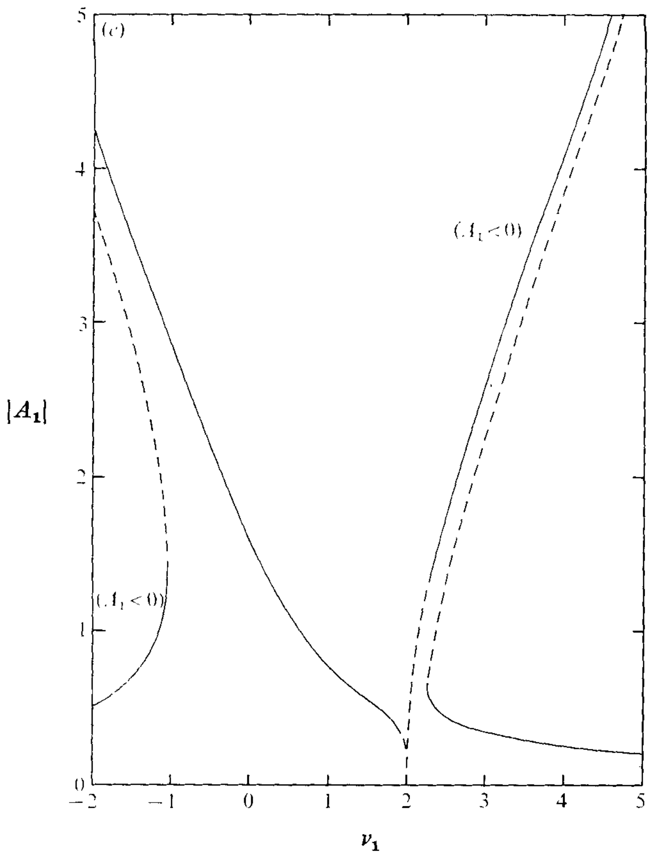


FIGURE 5. The response curve for forced harmonic oscillations, as calculated from (8.9). The dashed portions of the curves, over which (8.14) is violated, represent unstable motions. The upper and lower branches are the shoulders of two tilted resonant peaks, which are rendered finite by damping (see figure 7). (*a*) $\delta = 0$. (*b*) $\delta = 1$. (*c*) $\delta = 2$.

The response curves calculated from (8.9) for $\delta = 0, 1, 2$ are plotted in figure 5. The corresponding results for $\delta < 0$ are obtained by changing the signs of ν_1, A_1 and A_2. The critical values of ν_1, say $\nu_c^{(\pm)}$, between which (8.8) yields only one real value of A_1, are given by the two real roots of

$$\nu^4 - \delta\nu^3 - (\tfrac{3}{2})^3 = 0, \quad \nu = \nu_c^{(\pm)}(\delta) = -\nu_c^{(\mp)}(-\delta) \gtrless 0; \tag{8.10a, b}$$

see figure 6. The corresponding values of A_1 and A_2 are

$$A_{1c} = \tfrac{3}{2}\nu_c^{-1}, \quad A_{2c} = -\tfrac{1}{3}\nu_c \quad (\nu_1 = \nu_c). \tag{8.11a, b}$$

It remains to consider the stability of the preceding solutions. Substituting small perturbations of the form

$$C_n = A_n + C_n^{(1)}(\tau), \quad S_n = S_n^{(1)}(\tau) \tag{8.12a, b}$$

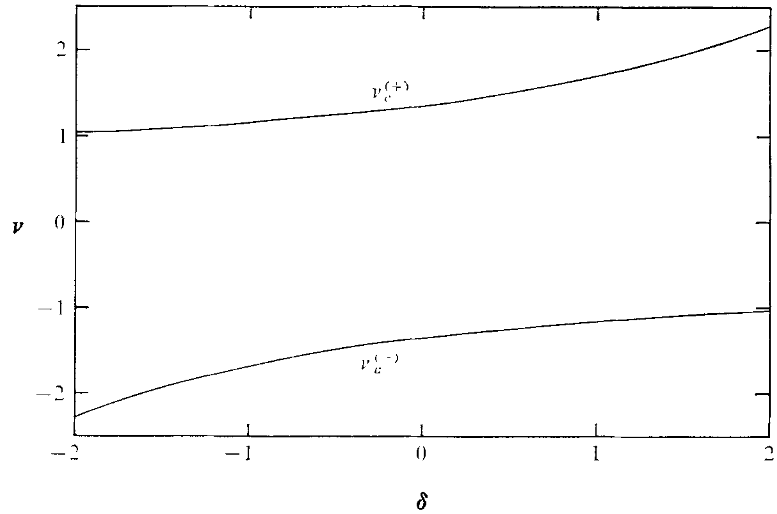


FIGURE 6. The critical values of ν_1, $\nu_c^{(\pm)} \gtrless 0$, between which the harmonic response $A_1(\nu_1)$ is single-valued; see (8.10).

into (8.5) with $\alpha_n \equiv 0$ therein, invoking (8.8), linearizing in $C_n^{(1)}$ and $S_n^{(1)}$, and requiring the characteristic determinant of the resulting homogeneous equations to vanish yields

$$\begin{vmatrix} \lambda & -\nu_1+A_2 & 0 & -A_1 \\ \nu_1+A_2 & \lambda & A_1 & 0 \\ 0 & -2A_1 & \lambda & -2\nu_2 \\ 2A_1 & 0 & 2\nu_2 & \lambda \end{vmatrix} = 0 \quad (\lambda \equiv d/d\tau). \tag{8.13}$$

Expanding the determinant and simplifying with the aid of (8.8) yields a quadratic equation for λ^2, both roots of which must be negative if the perturbation is to be stable. It follows from this requirement that necessary and sufficient conditions for stability are

$$4(\nu_1-\delta)(3\nu_1-\delta)A_1^2+2(4\delta-3\nu_1)A_1-1 > 4|(\nu_1-\delta)A_1|(3-2\nu_1A_1)^{\frac{1}{2}} > 0. \tag{8.14}$$

Those portions of the response curves in figure 5 on which (8.14) is not satisfied are dashed. We remark that motions that correspond to points on the upper branches, although stable, may be difficult to excite; moreover the results are restricted by $\nu_1 = O(1)$.

Harmonic solutions with damping

We now consider the effects of damping on harmonic response with

$$\nu_1 = \nu_2 \equiv \nu \quad (\delta = 0) \quad \text{and} \quad \alpha_1 = \alpha_2 \equiv \alpha;$$

the results for more general parametric combinations are asymmetric in ν and algebraically more complicated, but otherwise similar. Setting $\dot{C}_n = \dot{S}_n = 0$ in (8.5) and rearranging yields

$$(\alpha - S_2) C_1 - (\nu - C_2) S_1 = 0, \tag{8.15a}$$

$$(\nu + C_2) C_1 + (\alpha + S_2) S_1 = 1, \tag{8.15b}$$

$$2\alpha C_2 - 2\nu S_2 = 2S_1 C_1, \tag{8.15c}$$

$$2\nu C_2 + 2\alpha S_2 = S_1^2 - C_1^2. \tag{8.15d}$$

Solving (8.15c, d) for S_2 and C_2 and substituting the results into (8.15a, b) yields, after some algebraic reduction,

$$C_2 = \rho^2[\alpha S_1 C_1 + \tfrac{1}{2}\nu(S_1^2 - C_1^2)], \quad S_2 = \rho^2[-\nu S_1 C_1 + \tfrac{1}{2}\alpha(S_1^2 - C_1^2)], \tag{8.16a, b}$$

$$C_1 = \nu A_1^2(1 - \tfrac{1}{2}\rho^2 A_1^2), \quad S_1 = \alpha A_1^2(1 + \tfrac{1}{2}\rho^2 A_1^2), \tag{8.17a,b}$$

and

$$A_1^2[\alpha^2(1 + \tfrac{1}{2}\rho^2 A_1^2)^2 + \nu^2(1 - \tfrac{1}{2}\rho^2 A_1^2)^2] = 1, \tag{8.18}$$

where

$$A_1^2 = C_1^2 + S_1^2, \quad \rho^2 = (\alpha^2 + \nu^2)^{-1}. \tag{8.19}$$

Solving (8.18) as a quadratic in ν^2 yields

$$\nu^2 = -\alpha^2 + \tfrac{1}{2}(A_1^2 + A_1^{-2}) \pm \tfrac{1}{2}(2 - 8\alpha^2 A_1^2 + A_1^{-4})^{\frac{1}{2}}. \tag{8.20}$$

Substituting $\nu > 0$ into (8.17) and the resulting expressions for C_1 and S_1 into (8.16) yields C_n and S_n as functions of A_1. C_n and S_n are, respectively, odd and even functions of ν. Typical response curves given by (8.20) are plotted in figure 7. A_1^2 is a monotonically decreasing function of ν^2 if $\alpha > 1{\cdot}085$.

Considering small perturbations with respect to the solution determined by (8.16)–(8.18) and (8.20) leads to the characteristic equation [cf. (8.13)]

$$\begin{vmatrix} \lambda + \alpha - S_2 & -\nu + C_2 & S_1 & -C_1 \\ \nu + C_2 & \lambda + \alpha + S_2 & C_1 & S_1 \\ -2S_1 & -2C_1 & \lambda + 2\alpha & -2\nu \\ 2C_1 & -2S_1 & 2\nu & \lambda + 2\alpha \end{vmatrix} = 0. \tag{8.21}$$

Stability requires $\mathscr{R}\lambda > 0$ for each of the four roots of (8.21). No stable harmonic motion is possible in the frequency interval $0 \leqslant \nu < \nu_A$ if $\alpha < \alpha_A = 0{\cdot}319$ (ν_A decreases from ∞ to 0 as α increases from 0 to α_A). Two stable, and one unstable, harmonic motions are possible for $\nu_B < \nu < \nu_C$ if $\alpha < \alpha_B = 0{\cdot}328$. A single, stable, harmonic solution exists for all ν if $\alpha > \alpha_B$. These regimes are illustrated in figure 8 for $\alpha = \frac{1}{4}$ ($\nu_{A,B,C} = 0{\cdot}225, 1{\cdot}291, 1{\cdot}480$).

Anharmonic solutions

The differential equations (8.5) may be rewritten in the form

$$\dot{\mathbf{x}}(\tau) + \mathbf{f}(\mathbf{x}) = \{0, 1, 0, 0\}, \quad \mathbf{x} = \{C_1, S_1, C_2, S_2\}, \tag{8.22}$$

where $\mathbf{x}$ is a vector in a four-dimensional phase space and $\mathbf{f}$ is a quadratic function of $\mathbf{x}$. Forming the scalar product of $\mathbf{x}$ and $\dot{\mathbf{x}}$ yields (after some algebraic manipulation)

$$d(\tfrac{1}{2}|\mathbf{x}|^2)/d\tau = \alpha_1[S_0^2 - C_1^2 - (S_1 - S_0)^2] - 2\alpha_2(C_2^2 + S_2^2), \tag{8.23a}$$

where

$$S_0 = \mu/2\alpha_1. \tag{8.23b}$$

It follows from (8.23) that $\mathbf{x}$ is bounded and that the maximum possible equilibrium amplitude is $\mathbf{x}_m = 2\{0, S_0, 0, 0\}$. Moreover, the solution of (8.22) for

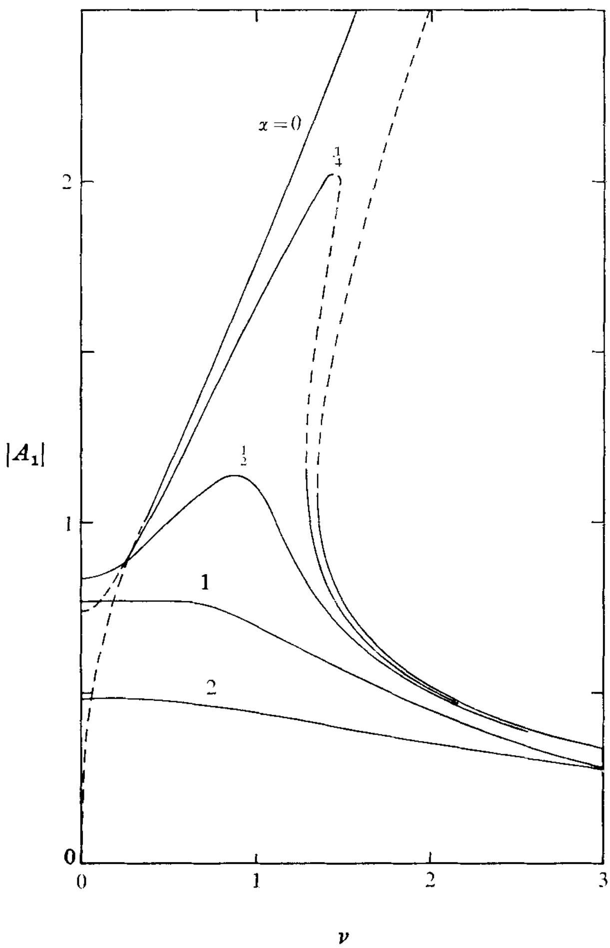


FIGURE 7. Response curve for damped harmonic oscillations for $\delta = 0$ and $\alpha_1 = \alpha_2 = \alpha$, as calculated from (8.20). The dashed portions of the curves represent unstable motions. The response is symmetric with respect to $\nu = 0$ (cf. figure 5*a*).

a prescribed initial vector $\mathbf{x}_0$ is unique by virtue of the fact that $\mathbf{f}$ is an integral function of $\mathbf{x}$.

The singular points of (8.22), $\mathbf{x}_s$ at which $\mathbf{f}(\mathbf{x}_s) = 0$, correspond to harmonic motions. Heuristic reasoning, supported by numerical integration (see figure 9), suggests that the trajectory from any $\mathbf{x}_0$ tends to a stable singular point if one exists. We consider further only the special case $\nu_1 = \nu_2 = \nu$, $\alpha_1 = \alpha_2 = \alpha$, for which four possibilities may be distinguished:

$$A: \alpha < \alpha_A, \quad 0 \leqslant \nu < \nu_A;$$

$$AB: \alpha < \alpha_B, \quad \nu_A < \nu < \nu_B \quad (\nu_A \equiv 0 \quad \text{if} \quad \alpha > \alpha_A);$$

$$BC: \alpha < \alpha_B, \quad \nu_B < \nu < \nu_C;$$

$$C: \nu > \nu_C \quad (\nu_C \equiv \nu_B \quad \text{if} \quad \alpha > \alpha_B).$$

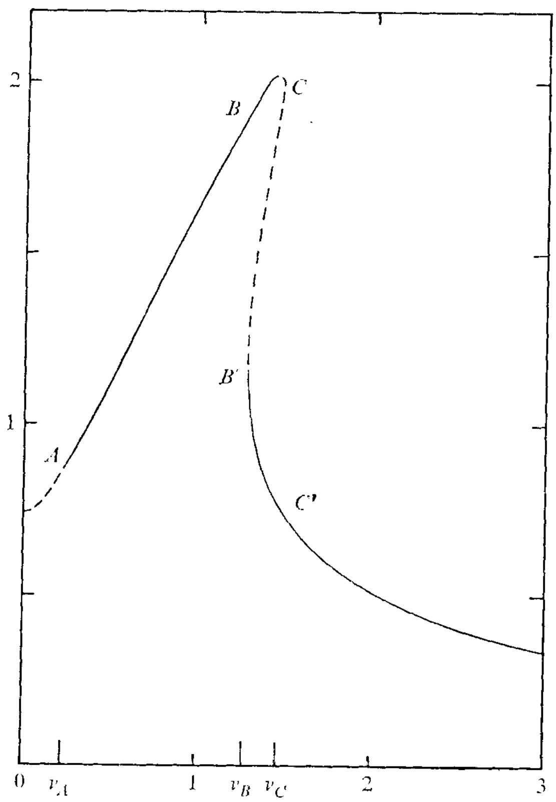
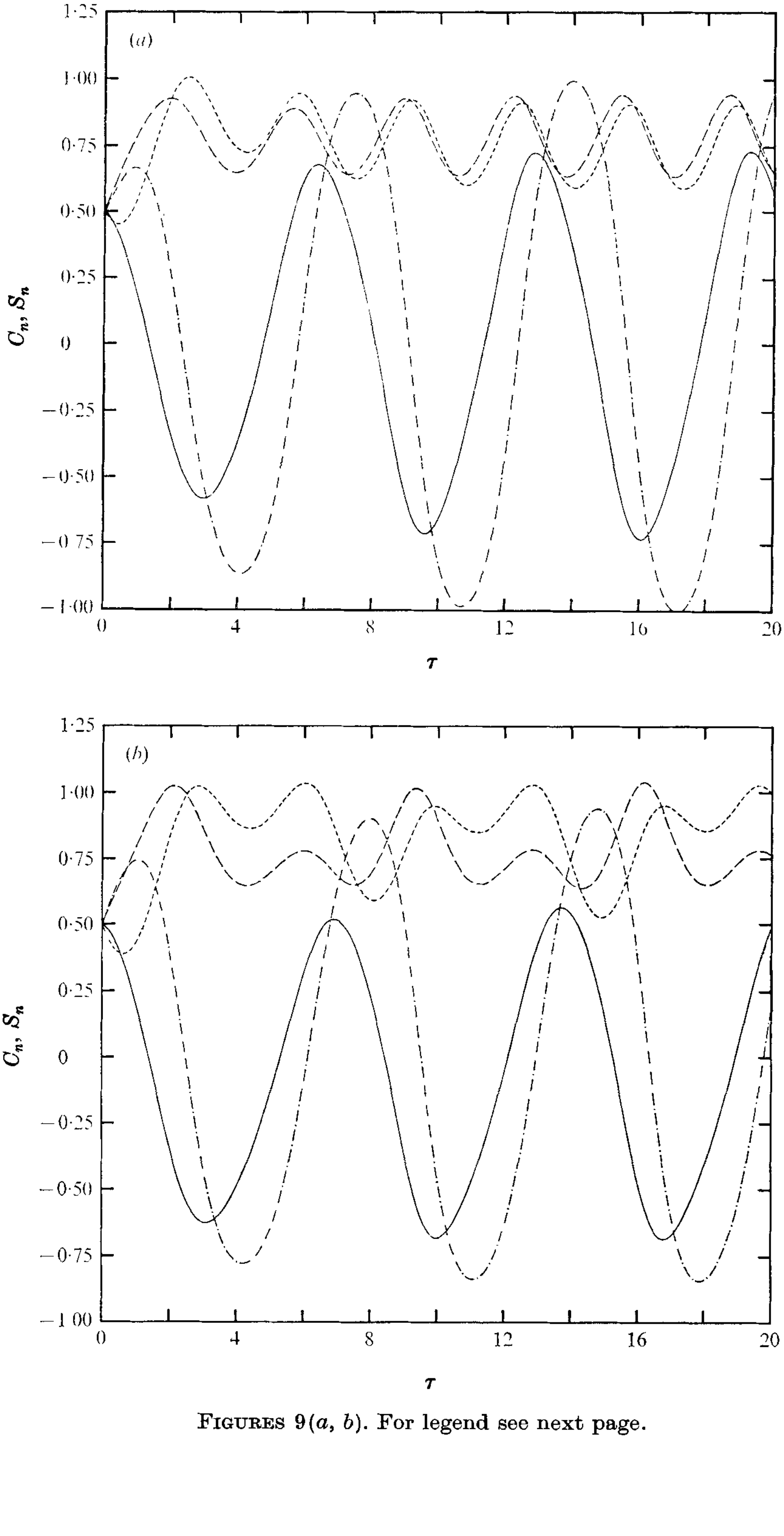


FIGURE 8. The regimes of stable (—) and unstable (- - -) harmonic motion described in §8 for $\alpha = \frac{1}{4}$ ($\nu_{A,B,C}$ = 0·225, 1·291, 1·480).

There is no stable singular point, but a limit cycle may exist, for A; see figures 9(a) and (b). There is one and only one singular point for either AB or C; see figure 9(c). There are two stable singular points for BC, and the asymptotic limit then depends on $\mathbf{x}_0$; see figure 9(d), for which the trajectory terminates on the stable singular point of lower energy, corresponding to the branch $B'C'$ in figure 8. Trajectories terminating at the stable singular point of higher energy, corresponding to the branch BC in figure 8, also were obtained by numerical integration, but only for initial conditions rather close to that singular point (such that the graphs of C_1, S_1, C_2 and S_2 *vs.* τ are almost flat).

The only analytical solutions of (8.23) that appear to merit further consideration are the limit cycles for $0 \leqslant \nu < \nu_A$.† These asymptotic solutions (we assume) may be constructed as Fourier series, with the fundamental period (for τ, not t) to be determined as part of the solution. We proceed to obtain the dominant terms

† It would of course, be desirable to have an analytical description of the $\mathbf{x}_0$ capture domains of the two stable singularities in regime BC, but this appears to be a difficult task.



FIGURES 9(a, b). For legend see next page.

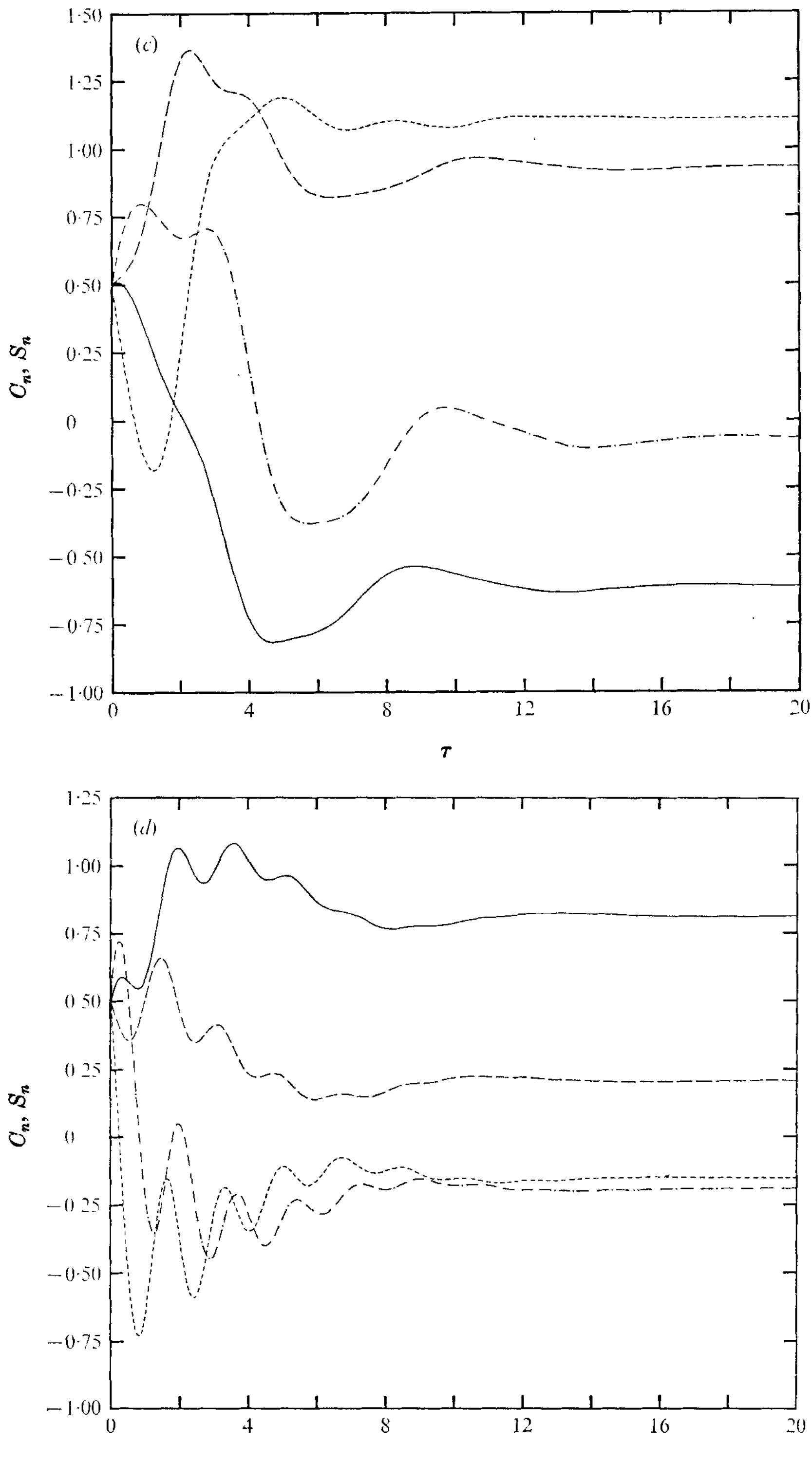


FIGURE 9. The numerical solution of (8.5) for $\alpha = \frac{1}{4}$ and (a) $\nu = 0$, (b) $\nu = 0{\cdot}1$, (c) $\nu = 0{\cdot}5$ and (d) $\nu = 1{\cdot}4$. ——, C_1; – – –, S_1; – · –, C_2; · · · ·, S_2.

for the simplest case: $\nu_1 = \nu_2 = 0$ ($\delta = \nu = 0$) and $\alpha_1 = \alpha_2 = \alpha$ (see figure 9*a*), for which (8.5) reduce to (after some re-arrangement)

$$\dot{C}_1 + (\alpha - S_2) C_1 + S_1 C_2 = 0, \tag{8.24a}$$

$$\dot{C}_2 + 2\alpha C_2 - 2S_1 C_1 = 0, \tag{8.24b}$$

$$\dot{S}_1 + \alpha S_1 + S_1 S_2 = 1 - C_1 C_2, \tag{8.24c}$$

$$\dot{S}_2 + 2\alpha S_2 - S_1^2 = -C_1^2. \tag{8.24d}$$

Inspection of (8.24) suggests an asymptotic solution in the form (we use *asymptotic* only to describe the limit $\tau \to \infty$, rather than in the more technical sense of an asymptotic series)

$$S_n(\tau) \sim S_{n0} + S_{n2} \cos(2\beta\tau + \phi_{n2}) + \dots \tag{8.25a}$$

and

$$C_n(\tau) \sim C_{n1} \cos(\beta\tau + \psi_{n1}) + \dots \quad (n = 1, 2). \tag{8.25b}$$

The frequency parameter β is unrelated to the parameter defined by (7.13), which does not appear in this section. Substituting the (by hypothesis) first approximations $S_n = S_{n0}$ into (8.24*a*, *b*), choosing $\psi_1 \equiv 0$ (one phase constant in the asymptotic solution may be chosen arbitrarily, corresponding to an arbitrary origin of τ), and equating coefficients of $\cos\beta\tau$ and $\sin\beta\tau$ yields (after some reduction)

$$C_{21} = 2^{\frac{1}{2}} C_{11}, \quad \psi_{21} = -\tan^{-1}(\beta/2\alpha), \tag{8.26a, b}$$

$$S_{10} = (2\alpha^2 + \tfrac{1}{2}\beta^2)^{\frac{1}{2}}, \quad S_{20} = 3\alpha. \tag{8.27a, b}$$

Substituting (8.25) into (8.24*c*, *d*), equating coefficients of $\cos(m\beta\tau)$ and $\sin(m\beta\tau)$ for $m = 0, 2$, and invoking (8.26) and (8.27) yields

$$C_{11} = (\beta^2 - 8\alpha^2)^{\frac{1}{2}}, \quad \beta = (6\alpha)^{-1}[1 + (1 + 288\alpha^4)^{\frac{1}{2}}]^{\frac{1}{2}}, \tag{8.28a, b}$$

$$\begin{bmatrix} 4\alpha & 2\beta & S_{10} & 0 \\ -2\beta & 4\alpha & 0 & S_{10} \\ -2S_{10} & 0 & 2\alpha & 2\beta \\ 0 & -2S_{10} & -2\beta & 2\alpha \end{bmatrix} \begin{Bmatrix} S_{12}\cos\phi_{12} \\ -S_{12}\sin\phi_{12} \\ S_{22}\cos\phi_{22} \\ -S_{22}\sin\phi_{22} \end{Bmatrix} = \frac{-C_{11}^2}{2S_{10}} \begin{Bmatrix} 2\alpha \\ \beta \\ S_{10} \\ 0 \end{Bmatrix}. \tag{8.29}$$

Higher approximations may be obtained by including additional terms in the Fourier series (8.25), but the algebraic complexity is formidable. The corresponding solutions for $0 < \nu < \nu_A$ contain the additional terms $S_{n1}\cos(\beta\tau + \phi_{n1})$ and C_{n0} in (8.25*a*) and (8.25*b*), respectively (cf. figure 9*b*).

Setting $\alpha = \frac{1}{4}$ in (8.26)–(8.28) yields $\beta = 1{\cdot}045$ (1·02), $S_{10} = 0{\cdot}819$ (0·794), $S_{20} = 0{\cdot}750$ (0·753), $C_{11} = 0{\cdot}770$ (0·734), $C_{21} = 1{\cdot}088$ (1·002), $S_{12} = 0{\cdot}165$ (0·156), and $S_{22} = 0{\cdot}147$ (0·159); the values in parentheses are provided by numerical integration (see figure 9*a*). The differences between these two sets of numbers presumably reflect the effects of higher harmonics in (8.25).

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Appendix. Development of $\mathscr{d}_{mn}$ and $\mathscr{k}_{mn}$

Substituting (2.7) into (2.9*a*), expanding the integrand in powers of η, and invoking (2.4*a*) for η yields (the summation convention does not apply anywhere in this appendix)

$$\mathscr{d}_{mn} = (S\cosh k_n d)^{-1}\iint \psi_m\psi_n \cosh k_n(d+\eta)\,dS \tag{A 1a}$$

$$= S^{-1}\iint \psi_m\psi_n(1+k_nT_n\eta+\tfrac{1}{2}k_n^2\eta^2+\ldots)\,dS \tag{A 1b}$$

$$= \delta_{mn}+k_nT_nC_{lmn}q_l+\tfrac{1}{2}k_n^2C_{jlmn}q_jq_l+\ldots, \tag{A 1c}$$

where $\quad C_{lmn} = S^{-1}\iint \psi_l\psi_m\psi_n\,dS, \quad C_{jlmn} = S^{-1}\iint \psi_j\psi_l\psi_m\psi_n\,dS, \ldots, \quad$ (A 2*a*, *b*)

and $$T_n = \tanh k_n d. \tag{A 3}$$

The corresponding expansion of (2.9*b*) yields

$$\begin{aligned}\mathscr{k}_{mn} &= (S\cosh k_m d\cosh k_n d)^{-1}\iint dS\int_{-d}^{\eta}[\nabla\psi_m.\nabla\psi_n\cosh k_m(y+d)\cosh k_n(y+d)\\ &\qquad + k_mk_n\psi_m\psi_n\sinh k_m(y+d)\sinh k_n(y+d)]\,dy \qquad \text{(A 4a)}\\ &= [2S(k_m\pm k_n)\cosh k_m d\cosh k_n d]^{-1}\iint(\nabla\psi_m.\nabla\psi_n\pm k_mk_n\psi_m\psi_n)\\ &\qquad \times\sinh[(k_m\pm k_n)(d+\eta)]\,dS \qquad \text{(A 4b)}\\ &= (2S)^{-1}\iint(\nabla\psi_m.\nabla\psi_n\pm k_mk_n\psi_m\psi_n)[(k_m\pm k_n)^{-1}(T_m\pm T_n)\\ &\qquad +(1\pm T_mT_n)\eta+\tfrac{1}{2}(k_m\pm k_n)(T_m\pm T_n)\eta^2+\ldots]\,dS \qquad \text{(A 4c)}\\ &= \tfrac{1}{2}(D_{mn}\pm k_mk_n\delta_{mn})(k_m\pm k_n)^{-1}(T_m\pm T_n)+\tfrac{1}{2}(D_{lmn}\pm k_mk_nC_{lmn})(1\pm T_mT_n)q_l\\ &\qquad +\tfrac{1}{4}(D_{jlmn}\pm k_mk_nC_{jlmn})(k_m\pm k_n)(T_m\pm T_n)q_jq_l+\ldots, \qquad \text{(A 4d)}\end{aligned}$$

where the sums of the alternatives with vertically ordered signs are implicit, and

$$D_{mn} = S^{-1}\iint\nabla\psi_m.\nabla\psi_n\,dS, \quad D_{lmn} = S^{-1}\iint\psi_l\nabla\psi_m.\nabla\psi_n\,dS, \ldots. \tag{A 5}$$

Invoking Green's first theorem, (2.5*a*, *b*) and (2.6) yields

$$D_{mn} = -S^{-1}\iint\psi_m\nabla^2\psi_n\,dS = \delta_{mn}k_n^2. \tag{A 6}$$

Similarly,

$$D_{lmn} = -S^{-1}\iint\psi_n(\psi_l\nabla^2\psi_m+\nabla\psi_l.\nabla\psi_m)\,dS = k_m^2C_{lmn}-D_{nlm}, \tag{A 7}$$

which may be iterated to obtain

$$D_{lmn} = \tfrac{1}{2}C_{lmn}(k_m^2+k_n^2-k_l^2), \tag{A 8}$$

and $$\begin{aligned}D_{jlmn} &= -\iint\psi_n(\psi_j\psi_l\nabla^2\psi_m+\psi_j\nabla\psi_l.\nabla\psi_m+\psi_l\nabla\psi_j.\nabla\psi_m)\,dS\\ &= k_m^2C_{jlmn}-D_{njlm}-D_{nljm},\end{aligned} \tag{A 9}$$

which provides useful results (e.g. $D_{nnnn} = \frac{1}{3}C_{nnnn}k_n^2$) but does not lead to a general result like (A 8).

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